Mean-Based Efficient Hybrid Numerical Method For Solving First-Order Ordinary Differential Equations

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Abstract

The center of interest of this work is to introduce a mean-based efficient hybrid numerical method for solving first-order ordinary differential equations with initial conditions. This method is generated by the mean of the modified slope of modified Euler's method and the mean of main incremental functions of RK-2 method and a second-order contra harmonic mean method. The factors related to the numerical method have been analyzed to observe its performance and found that the method satisfies all its factors. It also performs with better accuracy than modified Euler's methods, a second stage second-order contra harmonic mean method, and RK-2 method. The proposed method performed with better accuracy than the existing methods, but it is also better workable on some other second-order well-known methods.

Keywords—Mean-based iterative method; Initial value problems; Euler's methods; Contra harmonic mean method; RK-2 method

1 Introduction

Differential equations have a great scope in this modern world and perform a dynamic role in mathematical demonstrating [1]. Many application problems are related to engineering and science, and various differential equations could be generated [2-4]. These equations are categorized into ordinary and partial differential equations [5,6]. The solution of every differential equation cannot be obtained by using any general method [7]. So, these differential equations are further classified into different classes and again, each class is subdivided by the particular type of differential equations that have numerous numerical methods for obtaining its solutions [8]. The numerical methods are the best tools for solving differential equations whose analytical solution is tough [9]. Most of the physical phenomenon problems carried out in the fields of biology, chemistry, medicine, population dynamics, social sciences, and genetic engineering appear in the general form of

$$v' = h(\mu, v);\ v(\mu_0) = v_0 \tag{1}$$

which are solved by various researchers and scholars by generating different numerical methods[10-13]. The above general form of the differential equations is given as an initial value problem. The main focus of this research is to develop a new numerical method for solving initial value problems like eq. (1).

2 Concept and Methods

Many physical engineering and science problems can be solved by using numerical methods [14]. Among these methods, the oldest and most straightforward method is the first-order explicit linear iterative method which Leonhard Euler gave in 1768. That is given by (2).

$$v_{i+1} = v_i + \alpha h(\mu, v_i) \tag{2}$$

where α is the step size. The value v_{i+1} of Euler's method will be computed at the point (μ_i, v_i). Another Euler's 1^{st} order implicit method that is given by (3).

$$v_{i+1} = v_i + \alpha h(\mu_{i+1}, v_{i+1}) \tag{3}$$

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On expanding eq. (3) becomes

$$v_{i+1} = v_i + \alpha h(\mu_i + \alpha, v_i + \alpha h(\mu_i, v_i)) \tag{4}$$

Many researchers have worked on Euler's methods and made many modifications. Some of them are given by eq. (5), eq. (6) and eq. (7) [12] respectively.

$$v_{i+1} = v_i + \alpha h(\mu_i + \frac{\alpha}{2}, v_i + \frac{\alpha}{2}\ h(\mu_i, v_i)) \tag{5}$$

$$v_{i+1} = v_i + \frac{\alpha}{2}\ h[h(\mu_i, v_i) + h(\mu_i + \alpha, v_i + \alpha h(\mu_i, v_i))] \tag{6}$$

$$v_{i+1} = v_i + \alpha\ h[\mu_i + \frac{\alpha}{2}, v_i + \frac{\alpha}{2}\ h(\mu_i, v_i + \frac{\alpha}{2}\ h(\mu_i, v_i)] \tag{7}$$

Runge has developed different methods of different orders for solving initial value problems (IVPs), one of them is second-order Runge –Kutta (RK-2) [15], which is

$$v_{i+1} = v_i + \alpha\rho \tag{8}$$

where

$$\rho = \frac{\rho_1 + \rho_2}{2} \tag{9}$$

and $\rho_1 = h(\mu_i, v_i)$ $\rho_2 = h(\mu_i + \alpha, v_i + \alpha h(\mu_i, v_i))$ Butcher (1987) worked on Contra harmonic mean methods (C_0M). The second stage of second order (C_0M) method [16] given by

$$v_{i+1} = v_i + \alpha[\frac{\rho_1^2 + \rho_2^2}{\rho_1 + \rho_2}] \tag{10}$$

where $\rho_1 = h(\mu_i, v_i)$; $\rho_2 = h(\mu_i + \alpha, v_i + \alpha\ h(\mu_i, v_i))$ Eq. (10) can be written as

$$v_{i+1} = vi + \alpha R \tag{11}$$

$$whereR = [\frac{\rho_1^2 + \rho 2_2^2}{\rho 1 + \rho 2}] \tag{12}$$

and $\rho_1 = h(\mu_i, v_i)$; $\rho_2 = h(\mu_i + \alpha, v_i + \alpha h(\mu_i, v_i))$

3 Material and Method

For developing a new method, compare the inner slope of modified Euler's method (ME) eq. (5) with the main incremental function (R) of eq. (11) then obtained the following.

$$v_{i+1} = v_i + \alpha h(\mu_i + \frac{\alpha}{2}, v_i + \frac{\alpha}{2}[\frac{\rho_1^2 + \rho_2^2}{\rho_1 + \rho_2}]) \tag{13}$$

Eq. (13) also can be described as

$$v_{i+1} = v_i + \alpha\rho^* \tag{14}$$

where

$$\rho^* = h(\mu_i + \frac{\alpha}{2}, v_i + \frac{\alpha}{2}[\frac{\rho_1^2 + \rho_2^2}{\rho_1 + rho_2}]) \tag{15}$$

Hereby taking the mean of ρ and R of eq. (9) and eq. (12) respectively then has been found in eq. (16).

$$U = \frac{1}{2}[\rho + R] = \frac{1}{2}[\frac{\rho_1 + \rho_2}{2} + \frac{\rho_1^2 + \rho_2^2}{\rho_1 + \rho_2}] \tag{16}$$

Employed eq.(6) then compares $h(\mu_i, v_i)$ by ρ^* and $h(\mu_i + \alpha, v_i + \alpha h(\mu_i, v_i))$ by U of eq.(15) and eq. (16) respectively, then obtained as follows:

$$v_{i=1} = v_i + \frac{\alpha}{2}[\rho^* + U] \tag{17}$$

Substitute eq. (15) and eq. (16) into eq. (17) then has been obtained the following proposed method.

$$\begin{aligned} v_{i+1} = v_i + \frac{\alpha}{2}[h(\mu_i + \frac{\alpha}{2}, v_i + \frac{\alpha}{2}[\frac{\rho_1^2 + \rho_2^2}{\rho_1 + \rho_2}]) + \\ \frac{1}{2}[\frac{\rho_1 + \rho_2}{2} + \frac{\rho_1^2 + \rho_2^2}{\rho_1 + \rho_2}]] \end{aligned} \tag{18}$$

where $\rho_1 = h(\mu_i, v_i)$ and $\rho_2 = h(\mu_i + \alpha, v_i + \alpha h(\mu_i, v_i))$ problems related to eq. (1). For this numerical method; used slopes should not be initial zero at given initial conditions. Autonomous linear and non-linear initial value problems and non-autonomous linear and variables separable are best solved by this numerical method.

4 Stability Analysis

Stability analysis is an important factor of numerical methods for checking its performance. The stability analysis of any numerical method will be carried out by following the linear model problem.

$$v' = \lambda v, v(\mu_0) = v_0 \quad \mu_0 \leq \mu \tag{19}$$

where λ is a complex number[16-19]. Using eq. (18) in test problem eq.(19), and found eq. (20).

$$\begin{aligned} v_{i+1} = v_i + \frac{\alpha}{2}[\lambda(v_i + \frac{\alpha}{2}[\frac{(\lambda v_i)^2 + (\lambda(v_i + \alpha(\lambda v_i)))^2}{\lambda v_i + \lambda(v_i + \alpha\lambda v_1}]) + \\ \frac{1}{2}[\frac{2\lambda v_i + \alpha\lambda^2 v_i}{2} + \frac{2\lambda^2 v_i^2 + 2\alpha\lambda^3 v_i^2 + \alpha^2\lambda^4 v_i^2}{2\lambda v_i + \alpha\lambda^2 v_i}]] \\ \implies v_{i+1} = v_i + \frac{\alpha}{2}[\frac{2\lambda v_i + 2\alpha\lambda^2 v_i + \alpha^2\lambda^3 v_i + \alpha^3\frac{\lambda^4}{2}v_i}{2 + \alpha\lambda} \\ + \frac{2v_i + 2\alpha^2 v_i + \alpha^2\frac{\lambda^3}{4}v_i}{2 + \alpha\lambda}] \\ \implies v_{i+1} = (1 + \frac{2\alpha\lambda + 2\alpha^2\lambda^2 + \frac{5}{8}\alpha^3\lambda^3 + \frac{\alpha^4\lambda^4}{4}}{2 + \alpha\lambda})v_i \end{aligned} \tag{20}$$

Let $z = \lambda\alpha$, and substitute it in eq. (20) we obtain the stability function of the proposed method eq. (21).

$$\begin{aligned} \phi(z) = 1 + \frac{2z + 2z^2 + \frac{5}{8}z^3 + \frac{z^4}{4}}{2 + z} \\ \implies \phi(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{16}z^3 + \frac{3z^4}{16(2 + z)} \end{aligned} \tag{21}$$

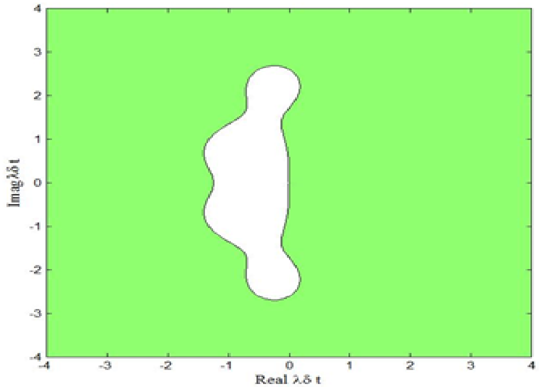

Fig. 1: Stability Region

The stability region of the developed method is shown in figure 1. The unshaded region of the figure expressed the stable region; therefore, the developed numerical method is stable.

5 Consistency Analysis

For the general form of eq. (1) can be described in the following general form of the iterative technique [3, 20-22].

$$v_{i+1} = v_i + \alpha\phi(\mu_i, v_i, \alpha) \tag{22}$$

It will be consistent with initial value problems if

$$\lim_{\alpha\to 0} \phi(\mu_i, v_i, \alpha) = h(\mu_i, v_i) \tag{23}$$

The main incremental function of eq. (18) is expressed as:

$$\phi(\mu_i, v_i, \alpha) = \frac{1}{2}[h(\mu_i + \frac{\alpha}{2}, v_i + \frac{\alpha}{2}[\frac{\rho_1^2+\rho_2^2}{\rho_1+\rho_2}]) + \frac{1}{2}[\frac{\rho_1+\rho 2}{2} + \frac{\rho_1^2+\rho_2^2}{\rho_1+\rho_2}]]$$

Now, using eq. (23) on the main incremental function of eq. (18) and found that it has been satisfied.

$$\lim_{\alpha\to 0} \phi(\mu_i, v_i, \alpha) = \lim_{\alpha\to 0} \frac{1}{2}[h(\mu_i + \frac{\alpha}{2}, v_i + \frac{\alpha}{2}[\frac{\rho_1^2+\rho_2^2}{\rho_1+\rho_2}] + \frac{1}{2}[\frac{\rho_1+\rho_2}{2} + \frac{\rho_1^+\rho_2^2}{\rho_1+\rho_2}]]$$

$$\lim_{\alpha\to 0} \phi(\mu_i, v_i, \alpha) = \lim_{\alpha\to 0} \frac{1}{2}[h(\mu_i, v_i) + \frac{1}{2}[h(\mu_i, v_i) + [\frac{2[h(\mu_i, v_i)]^2}{2h(\mu_i, v_i)}]]]$$

$$\lim_{\alpha\to 0} \phi(\mu_i, v_i, \alpha) = \lim_{\alpha\to 0} \frac{1}{2}[2h(\mu_i, v_i)]$$

$$\lim_{\alpha\to 0} \phi(\mu_i, v_i, \alpha) = h(\mu_i, v_i)$$

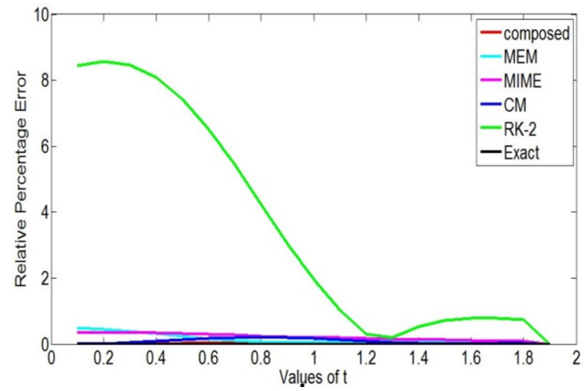

Fig. 2: Graph of Relative Percentage Error corresponding to different t-values

The developed method is consistent, whose proof is shown by the above derivation satisfies the eq condition. (22). It is enough evidence to assure that the developed method is consistent.

6 Results and Discussions

Here, the relative percentage error of some problems related to eq. (1) has been compared with a few existing and developed methods. The existing methods, which are taken for the comparison, are modified Euler's method (MEM), modified improved modified Euler's method (MIME), contra harmonic mean method (C_oM), and second order Runge-Kutta method. The following examples have been compared through the graphs of relative percentage error with corresponding t values.

***Example1* :** Solve [23] $y'(t) = 1 + 2y - y^2, y(0) = 0$ and $h = 0.1$ in the interval $[0, 2]$. The theoretical solution is
$y(t) = 1 + \sqrt{2}\ tan\ h[\sqrt{2}\ t + \frac{1}{2}\ln(\frac{\sqrt{2}-1}{\sqrt{2}+1})]$

The above figure shows the graphs of relative percentage error corresponding to different t-values of MEM, MIME, C0M, and RK-2 methods. As fig. 2 described, the relative percentage error of RK-2 method shows the fluctuation between t =1.2 and t =1.4, but other graphs converged more slowly than the composing method.

***Example2* :** Solve $y'(t) = 2y + 3e^t\ ; y_0 = 0\ at\ t_0 = 0$ in the interval [0, 10]. The analytical solution is $y(t) = 3(e^2t - e^t)$.

Fig. 3, the relative percentage error of the composing method converges faster than the relative percentage error of other methods. Therefore, it means that the composing method works better than the others.

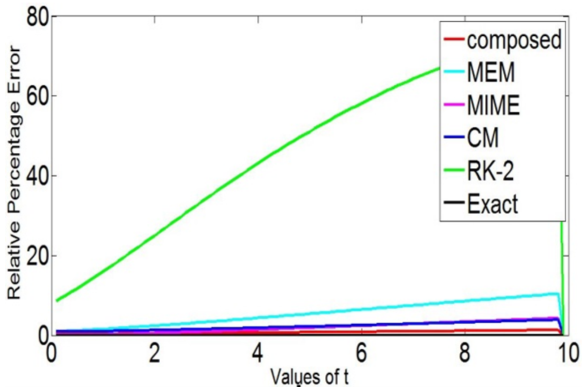

Fig. 3: Graphs of Relative Percentage Error corresponding to different t-values

7 Conclusion

This research work has been carried out to obtain the solution for the problems in eq. (1) by developing a new iterative method. The method has been developed by taking the mean of the modified slope of modified Euler's method and the mean of K of RK-2 and K of second order contra harmonic mean method. This method uses three slopes per integration step and other methods use two or three slopes per integration step. This method works better than the MEM, MIME, C_oM, and RK-2 methods, But it is also better than those methods, which evaluate two or three slopes per integration step and are of second order. The number of initial value problems has been tested for accuracy through the MATLAB software and found that the composing method performs better than the other methods. The developed method is also analyzed for some factors of the numerical method and observed that the composing method is consistent, stable, and accurate.

References

[1] Shaikh, Wajid Ali, Muhammad Usman Keerio, Abdul Ghafoor Shaikh, and Lubna Naz. "Higher Order Runge-Kutta Method for Solving System of Ordinary Differential Equations Using MS Excel." Quaid-E-Awam University Research Journal of Engineering, Science Technology, Nawabshah. 19, no. 2 (2021): 36-41.

[2] Kumar, Prem, Asif Ali Shaikh, and Sania Qureshi. "Non-Linear Numerical Schemes for Exact Solutions of Initial Value Problems." IJSRD-International Journal for Scientific Research Development 6, no. 07 (2018): 2321-0613.

[3] Memon, Z. U. N., M. S. Chandio, and S. Qureshi. "On consistency, stability and convergence of a modified ordinary differential equation solver." Sindh University Research Journal-SURJ (Science Series) 47, no. 4 (2015).

[4] Ochoche, Abraham. "Improving the improved modified Euler method for better performance on autonomous initial value problems." Leonardo Journal of Sciences 12 (2008): 57-66.

[5] Ram, Tulja, Muhammad Anwar Solangi, and Ali Asghar Sanghah. "A hybrid numerical method with greater efficiency for solving initial value problems." Mathematical Theory and Modeling 10, no. 2 (2020).

[6] Ahmad, Najmuddin, and Shiv Charan. "A Comparative Study on Numerical Solution of Ordinary Differential Equation By Different Method With Initial Value Problem." Int. J Recent Sci Res 8, no. 10 (2017): 21134-21139.

[7] Qureshi, S., Z. Memon, A. A. Shaikh, and M. S. Chandio. "On the Construction and Comparison of an Explicit Iterative Algorithm with Nonstandard Finite Difference Schemes." Mathematical Theory and Modeling 3, no. 13 (2013): 78-87.

[8] Chandio, M. S., and A. G. Memon. "MPROVING THE EFFICIENCY OF HEUN'S METHOD." Sindh University Research Journal-SURJ (Science Series) 42, no. 2 (2010).

[9] Shaikh, Abdul Hannan, Abdul Ghafoor Shaikh, Wajid Ali Shaikh, and Asif Ali. "Polynomial Finite Difference Technique to Generalize Helmholtz Type Equations." Quaid-E-Awam University Research Journal of Engineering, Science Technology, Nawabshah. 19, no. 2 (2021): 15-21.

[10] Ansari, Muhammad Yasir, Asif Ali Shaikh, and Sania Qureshi. "Error bounds for a numerical scheme with reduced slope evaluations." J. Appl. Environ. Biol. Sci 8, no. 7 (2018): 67-76.

[11] Tunio, Naveed Ahmed, Asif Ali Shaikh, and Sania Qureshi. "TO DEVELOP EFFICIENT SCHEME FOR SOLVING INITIAL VALUE PROBLEM IN ORDINARY DIFFERENTIAL EQUATION." (2019).

[12] CHANDI, SALI, AALI SHAIKH, and S. QURESHI. "A Modified Algorithm of Improved Explicit Euler's Method To Solve Initial Value Problems."

[13] Kamruzzaman, Md, and Mithun Chandra Nath. "A comparative study on numerical solution of initial value problem by using Euler's method, modified Euler's method and runge–kutta method." Journal of Computer and Mathematical Sciences 9, no. 5 (2018): 493-500.

[14] Shaikh, W. A., A. G. Shaikh, M. Memon, A. H. Sheikh, and A. A. Shaikh. "NUMERICAL HYBRID ITERATIVE TECHNIQUE FOR SOLVING NONLINEAR EQUATIONS IN ONE VARIABLE."

[15] [15] Shaikh,A.G., Keerio,U., Wajid ,S. Sheikh, S. K (2022), "Numetrical higher order Runge-Kutta methods in transient and dumping analysis", international journal of advance and applied sciences, (Technical note)9(10),pp-174-179.

[16] Yaakub, A. R., and David J. Evans. "A fourth order Runge–Kutta RK (4, 4) method with error control." International journal of computer mathematics 71, no. 3 (1999): 383-411.

[17] Qureshi, S., A. A. Shaikh, Z.U.N. Memon, and M.S. Chandio, (2014), "Analysis of Accuracy, Stability, Consistency and Convergence of an Explicit Iterative Algorithm. Mathematical Theory and Modeling", 4 (1): 1-10.

[18] Akanbi, Moses A. "Third order Euler method for numerical solution of ordinary differential equations." ARPN J. Eng. Appl. Sci 5 (2010): 42-49.

[19] Qureshi, S., A. A. Shaikh, and M. S. Chandio. "Critical study of a nonlinear numerical scheme for initial value problems." Sindh University Research Journal-SURJ (Science Series) 47, no. 4 (2015).

[20] Aliya, Tasneem, A. Ali Shaikh, and Sania Qureshi. "Development of a nonlinear hybrid numerical method." Advances in Differential Equations and Control Processes 19, no. 3 (2018): 275-285.

[21] Kandhro, Muhammad Daud, Asif Ali Shaikh, and Sania Qureshi. "A new third order iterative integrator for Cauchy problems." J. Appl. Environ. Biol. Sci 8, no. 7 (2018): 26-35.
[22] Shaikh, A. G., Wajid Shaikh, A. H. Shaikh, and Muhammad Memon. "Advanced efficient iterative methods to the Helmholtz equation." (2022).
[23] Patra, A., T. T. Shone, and B. B. Mishra. "Natural decomposition approximation solution for first order nonlinear differential equations." International Journal of Engineering Technology 7, no. 4.5 (2018): 442-445.