# Numerical Simulation of One-Dimensional Advection Diffusion Equation by New Hybrid Explicit Finite Difference Schemes

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**Abstract**

In this paper, the aim is to develop two Hybrid Explicit Schemes based on the Finite Difference method for a one-dimensional Advection Diffusion Equation. Moreover, the study considered the advection-diffusion equation as an initial boundary value problem (IBVP) for numerical solutions obtained from various second-order explicit methods along with the solution by proposed methods. Von-Neumann stability analysis is used to analyze the stability of the developed schemes graphically. In the numerical analysis of errors, the L_2 has been computed to compare proposed methods with existing methods in literature and has been carried out in different conditions and step sizes. Proposed methods are robust explicit methods for the purpose of solution of the 1-D advection-diffusion equation.

**Keywords**—Advection Diffusion Equation, Explicit Schemes, Finite Difference Method, Hybrid, Von Neumann Stability Analysis

## 1 Introduction

The study of partial differential equations has been a major concern in the history of mathematics as well as in engineering so mathematical models is used in understanding of physical sciences. So, Advection Diffusion Equation (ADE) is also one of them and uses as model describing transport and diffusion problem. Using finite difference schemes explicit and implicit techniques such as Forward Time Center Space scheme (FTCS), Upwind, Lax-Wendroff for numerical approximation of 1 D (ADE), in same work author also describe accuracy, and stability of these schemes [1]. In numerical approximation of one-dimensional parabolic (PDE) with nonlocal time weighing initial conditions by developing a new modified explicit scheme based on finite difference method, authors modified of Saulyev's first kind formula with the aim of reducing error and finally compared the results with existing method such as FTCS, Duke-Frankel, Crandal's technique and Crank Nicholson's scheme in different conditions and step sizes also proved new modified scheme unconditionally stable [2]. The (ADE) was used as mathematical model to express transporting and diffusion problems, for water quality model in a canal computing through Saulyev's finite difference technique [3]. In this research, authors focused on the utilization of parallel architectures for implementing iterative methods in the context of solving large systems of equations, in the literature a problem widely studied solutions up to 12 cores using the Jacobi method, but authors analyzed by employing a virtual machine with 30 cores, first time used to find the solution of Helmholtz equation resulting in more comprehensive findings through their experimentation, and demonstrated effectiveness of parallel computing methodologies and their applicability and OpenMP implementations was on both current supercomputing platforms and virtual machine environments [4]. The (ADRE) was used as a model and applied the scheme to a pollutant dispersion and for water pollutant concentration also the graphical solutions is obtained [5]. Authors used the one-dimensional advection dispersion reaction equation (ADRE) as dispersion model using backward time central space scheme that gives to pollution concentration fields for elevation of the water flow [6]. Scholars take governing equation (ADRE) equation as model for uniform flow of canal water quality by using two modified explicit schemes (FTCS) and Saulyev's technique to compute pollutant concentration fields after evaluating velocity, also compared both schemes stability and accu-

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racy [7]. Authors investigated the convergence rate of proposed hybrid numerical iterative technique for solving nonlinear problems of one variable (f(x) = 0), hence various algebraic and transcendental nonlinear problems of one variable were solved which exhibited a convergence rate of two, indicating its efficiency in finding approximate real roots in [8] show casing its effectiveness in achieving accurate solutions. For obtaining solution, the MATLAB tool was used with implementing the iterative techniques. In [9], authors took a model based one-dimensional linear wave equation investigated and compared performance of these techniques using lax -Wendroff method and finite difference method is applied in discretization and many more which were more stable than single step methods also other techniques which are time consuming in numerical in accuracy. Authors discuss the numerical solution of Advection diffusion in one dimension for high Peclet numbers using five classical methods Forward Time Center Space, Backward Time Center Space, Crank Nicholson, MacCormak and Saulyev's. Also revised the latter two methods for study the pulse and step inputs of mass to a steady flow in a channel along with stability and accuracy [10]. Authors took (ADE) as model with two case studies and variable coefficients and two different sets of (BC) were considered at the inlet and outlet of the domain. For analytic solution Laplace transform is used, both analytical as well as numerical results were in good agreement with each other [11]. The focus of author is to study the stability and consistency analysis for (ADE) using the Central Difference Scheme (CDS) used Taylor's series expansion for (CDS) and found scheme is consistent also for stability Von-Neumann Method used for scheme and found to be conditionally stable [12]. For estimation of water pollution, the authors take one-dimensional (ADE) as an initial-boundary (IBVP) problems by finite difference methods also comparison with an exact solution, also represented solution graphically. Also, relative error is estimated [13]. In this paper author proposed one dimensional unsteady linear (ADE) was solved by both analytical and numerical methods in which Euler methods and Crank Nicholson method were used as numerical solutions while separation of variables method issued as analytical solution applied with homogenous (BC) and an (IC) in the form of a cosine function its stability depends on the viscosity term, exact solution simplified form which is confirmed by all numerical techniques [14]. For solving nonlinear convection-diffusion-reaction problems authors proposed hybrid iterative technique, referred to as the Variational Iteration Method, and proved as fast convergence in obtaining accurate solutions also comparative analysis was conducted against existing schemes, namely the variational iteration method, Through the comparison of error profiles, and examined accuracy and reliability of the proposed method, introduced a hybrid iterative technique, fusing the variational iteration method with the Chebyshev wavelet, The comparative analysis with existing methods further validates the proposed approach by showcasing its performance through error profiles [15]. For estimation of pollutant transport in a straight narrow channel, take (ADE) as mathematical model by using explicit upwind scheme also compare results numerically existing analytical solution finally results show good agreement [16]. Authors used (ADE) for numerical diffusion and oscillatory behavior characteristics are averted by two numerical methods semi-discretization and Euler's method were used for a system of (ODEs), finally compare the solutions and errors for both schemes [17]. In this paper authors presented numerical simulation as well as analytical solution of one-dimensional(ADE) by using explicit (CDS) and Crank-Nicolson scheme also developed a computer programming code for Crank-Nicolson technique and present the stability analysis for (ADE) for prescribed initial and boundary data, also error estimated, and the rate of convergence is graphically presented. Finally, using numerical schemes, pollutant in a river was estimated by authors with different times and points [18]. Authors took (ADE) with constant coefficients, for analytical solution Laplace transform was used and for numerical solution explicit finite difference schemes was used, to describe numerically and graphically the variation in pollutant concentration and dispersion (IC) and (BC) at the source of pollution (x = 0) were applied [19]. In medical field, for artificial blood oxygenation to aspect of cardiopulmonary bypass surgery, author take simple model and associated analytic solutions in order to maintaining physiological levels of oxygen and many more uses also new approach to enhancing the diffusion of oxygen into the blood artificially so for showing that using transverse flow [20]. Authors proposed A Numerical Hybrid Iterative Technique (NHIT) for estimating the real roots of nonlinear Eqs. In one v ariable (NLEOV) to accelerate the convergence of NLEOV solutions. Proposed method involved combining different methods to enhance performance and constructed by integrating the Taylor Series method with (N R) method in derivation. For computational analysis Excel and MATLAB tools were used of variety of NLEOV problems, and the results demonstrate superior convergence compared to the bracketing iterative method (BIM) also finally compared the results with existing schemes proved with improved convergence and accuracy [21]. The same authors take one dimensional (ADE) for describing the transport and diffusion problems for pollutants and suspended matter in a river by using Saulyev's scheme calculated at three-time steps $\theta = 0{,}0.5{,}1$ in which if $\theta = 0$ have glossy solution [22]. Authors used one-dimensional equations of conservation law form by using Saulyev's finite difference and computed the solution and compared it with the existing methods [23].

## 2 Mathematical Model

The model of advection diffusion equation in one dimensional form is as described by equations (1) to (4).

$$\frac{\delta u}{\delta t} + \beta\frac{\delta u}{\delta x} = \alpha\frac{\delta^2 u}{\delta x^2} \; where \; 0 < x < 1, 0 < t \leq T \quad (1)$$

With initial coditions

$$u(x,0) = f(x), 0 \leq x \leq 1, \quad (2)$$

And boundary conditions,

$$u(0,t) = b_0(t), 0 < t \leq T, \quad (3)$$

$$u(1,t) = b_1(t), 0 < t \leq T, \quad (4)$$

Where $f$, $b_0$ and $b_1$ are defined functions, while the function $u$ is unknown solution of governing equation and $\alpha, \beta > 0$ are chosen so that the process of diffusion and advection can be computed, respectively.

## 3 Existing Schemes

Various two levels of explicit and implicit schemes are present to solve the model problem (1-4). The FTCS, Upwind, Lax Wendroff [1] Saulyev's Type-I in [10] and Saulyev's Type-II in [3] are among explicit schemes, also (BTCS), upwind implicit formula, Crank–Nicolson, modified Siemieniuch–Gladwell procedure are implicit schemes as in [1]. We present some existing explicit and proposed Hybrid schemes in this section.

### 3.1 Forward Time and Center Space Type

The three schemes with forward difference quotient for time derivative approximation and first spatial derivative term with a weight $\phi$ and second order derivative term central difference approximation are [1] given as.

$$\frac{\delta u}{\delta t} = \frac{{u_i}^{n+1} - {u_i}^n}{\Delta t} \quad (5)$$

$$\frac{\delta u}{\delta x} = \phi\frac{({u_i}^n - u^n_{i-1})}{\Delta x} + (1-\phi)\frac{({u^n}_{i+1} - u^n_{i-1})}{2\Delta x} \quad (6)$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{u^n_{i+1} - 2{u_i}^n + u^n_{i-1}}{(\Delta x)^2} \quad (7)$$

Substituting $\phi = 0$, yields the following Forward time center space schemes,

$${u_i}^{n+1} = \frac{1}{2}(2s+c){u^n}_{i-1} + (1-2s)u_i + \frac{1}{2}(2s-c)u^n_{i+1} \quad (8)$$

where $c = \beta\frac{\Delta t}{\Delta x}$ , $s = \alpha\frac{\Delta t}{(\Delta x)^2}$ as in [1].

### 3.2 Upwind Type

From the approximation, the equations (5)-(7) with $\phi = 1$, yield the Lax-Wendroff explicit formula for approximation of unknown is

$$u_i^{n+1} = (s+c)u^n_{i-1} + (1-2s-c)u^n_i + su^n_{i+1} \quad (9)$$

### 3.3 Lax-Wendroff

From the approximation, the equations (5)-(7) with $\phi = c$, yield the upwind explicit formula for approximation of unknown is

$$u_i^{n+1} = \frac{1}{2}(2s+c+c^2)u^n_{i-1} + (1-2s-c^2)u^n_i + \frac{1}{2}(2s-c+c^2)u^n_{i+1} \quad (10)$$

### 3.4 Saulyev's Type-I

Using Saulyev's Type-I. The scheme uses forward difference quotient for time derivative as described in (5). The first order space derivative and second order space derivative terms are approximated as in [10] The first order space derivative term is approximated as

$$\frac{\delta u}{\delta x} = \frac{(u^n_{i+1} - u^{n+1}_{i-1})}{2\Delta x} \quad (11)$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{u^{n+1}_{i-1} - u^{n+1}_i - u^n_i + u^n_{i+1}}{(\Delta x)^2} \quad (12)$$

The computation formula for this scheme is described as,

$$u_i^{n+1} = (\frac{1}{1+s})[(s+\frac{c}{2})u^{n+1}_{i-1} + (1-s)u^n_i + (s-\frac{c}{2})u^n_{i+1}] \quad (13)$$

### 3.5 Saulyev's Type-II

On ADE using Saulyev's, in this work approximation of time derivative term is same as (5), but space derivatives are approximated as in [3],

$$\frac{\delta u}{\delta x} = \frac{1}{2}\frac{(u_{i+1}^n - u_i^n)}{\Delta x} + \frac{1}{2}\frac{(u_i^{n+1} - u_{i-1}^{n+1})}{\Delta x} \quad (14)$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{\theta}{(\Delta x)^2}(u_{i-1}^{n+1} - u_i^{n+1} - u_i^n + u_{i+1}^n) + \frac{(1-\theta)}{(\Delta x)^2} + (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad (15)$$

The computation formula for this scheme is,

$$u_i^{n+1} = \frac{1}{(1+\frac{c}{2}+s\theta)}[(\frac{c}{2}+s\theta)u_{i-1}^{n+1} + (s-\frac{c}{2})u_{i+1}^n + (1+\frac{c}{2}+s\theta-2s)u_i^n + s(1-\theta)u_{i-1}^n] \quad (16)$$

## 4 Proposed Hybrid Explicit Schemes

### 4.1 Hybrid Scheme 1

In this Proposed Hybrid Scheme (HS1), we use forward difference quotient for time derivative approximation as in (5) and first order spatial derivative term and second order derivative term are given as.

$$\frac{\delta u}{\delta x} = \phi\frac{(u_i^{n-1} - u_{i-1}^{n-1})}{\Delta x} + (1-\phi)\frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x} \quad (17)$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{\theta}{(\Delta x^2)}(u_{i-1}^{n+1}u_i^{n+1} - u_i^n + u_{i+1}^n) + \frac{(1-\theta)}{(\Delta x^2)}(u_{i+1}^n - 2{u_i}^n + u_{i-1}^n) \quad (18)$$

Now, taking for $\phi = c$ & $\theta = c$ then substituting (5), (17) and (18) in (1) yields the following,

$$u_i^{n+1} = (\frac{1}{1+sc})[(scu_{i-1}^{n+1} + (s(1-c)+\frac{c}{2}(1-c))u_{i-1}^n + (1-sc-2s(1-c))u_i^n + (sc+s(1-c)-\frac{c}{2}(1-c))u_{i+1}^n - c^2(u_i^{n-1} - u_{i-1}^{n-1}))] \quad (19)$$

### 4.2 Hybrid Scheme 2

In this Proposed Hybrid Scheme (HS2), we use forward difference quotient for time derivative approximation as in (5) and first order spatial derivative term $\phi = c$ in (6) and second order derivative term by $\theta = s^2$ in (15) are given as.

$$\frac{\delta u}{\delta x} = c\frac{(u_i^n - u_{i-1}^n)}{\Delta x} + (1-c)\frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x} \quad (20)$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{s^2}{(\Delta x^2)}(u_{i-1}^{n+1} - u_i^{n+1} - u_i^n + u_{i+1}^n) + \frac{(1-s^2)}{(\Delta x^2)}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad (21)$$

Now, substituting (5), (20) and (21) in (1) yields the following,

$$u_i^{n+1} = \frac{1}{1+s^3}[(\frac{c^2}{2}-\frac{c}{2}+s)u_{i+1}^n + (1-c^2-2s+s^3)u_i^n + (\frac{c^2}{2}+\frac{c}{2}+s-s^3)u_{i-1}^n + s^3u_{i-1}^{n+1}] \quad (22)$$

## 5 Numerical Problem

The Following numerical problem equations (23-26) have been taken from [1] and set with the model equations (1-4) for estimating the performance of the proposed schemes. The results of the proposed schemes are compared with the conventional schemes, as shown in the comparison table 1.

$$f(x) = exp(-\frac{(x+0.5)^2}{0.00125}) \quad (23)$$

$$g_0(0,t) = \frac{0.025}{\sqrt{0.000625+0.02t}}exp(-(\frac{(0.5-t)^2}{0.00125+0.04t})) \quad (24)$$

$$g_1(1,t) = \frac{0.025}{\sqrt{0.000625+0.02t}}exp(-(\frac{(1.5-t)^2}{0.00125+0.04t})) \quad (25)$$

With $\alpha = 0.01$ and $\beta = 0.1$ The exact solution is

$$g(x,t) = \frac{0.025}{\sqrt{0.000625+0.02t}}exp(-(\frac{(x+0.5-t)^2}{0.00125+0.04t})) \quad (26)$$

Note that: $L_2$ error norm is defined as

$$L_2 = ||\tilde{u} - u||_2 = \sqrt{\frac{1}{m}\sum_{i=1}^{m}|\tilde{u_l} - u_i|^2} \quad (27)$$

In (27), $\tilde{u_l}$ is exact solution and $u_i$ is approximate solution.

## 6 Results and Discussions

For the purpose of testing the $L_2$ Norm of the error has been computed through MATLAB tool at T = 1 with various Renold numbers (R) 1, 2, 4, 8, and 1.25, 2.5, and 5 respectively, along with values of different parameters c and s were set with step sizes to produce

TABLE 1: Performance Evaluation of the Performance Evaluation of the Proposed schemes with conventional methods at T=1

| R | c | s | M | N | FTCS | Upwind | Lax Wen-roff | Saul-yev's Type-1 | Saul-yev's Type-2 =0 (ST2-1) | Saul-yev's Type-2 =0.5 (ST2-2) | Saul-yev's Type-2 =1 (ST2-3) | Prop-osed Hybrid (HS1) | Prop-osed Hybrid (HS2) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0.05 | 0.025 | 50 | 1000 | 9.8124 e-03 | 9.4735 e-02 | 7.9606 e-03 | 3.6132 e-02 | 9.4046 e-03 | 1.0838 e-02 | 1.2923 e-02 | 7.8735 e-03 | 7.9619 e-03 |
| 2 | 0.1 | 0.05 | 50 | 500 | 1.4151 e-02 | 9.0675 e-02 | 6.4818 e-03 | 7.8137 e-02 | 9.4398 e-03 | 1.2949 e-02 | 1.8015 e-02 | 6.2554 e-03 | 6.4922 e-03 |
| 2 | 0.2 | 0.1 | 50 | 250 | 2.7158 e-02 | 8.2298 e-02 | 3.3824 e-03 | 1.6538 e-01 | 9.5810 e-03 | 1.8088 e-02 | 2.9110 e-02 | 5.5870 e-03 | 3.4812 e-03 |
| 2 | 0.4 | 0.2 | 50 | 125 | 5.9187 e-02 | 6.4550 e-02 | 3.4072 e-03 | 3.4525 e-01 | 1.0147 e-02 | 2.9287 e-02 | 5.0606 e-02 | 2.3213 e-02 | 3.4152 e-03 |
| 4 | 0.01 25 | 0.00 3125 | 25 | 2000 | 2.5417 e-02 | 1.1329 e-01 | 2.5314 e-02 | 2.2279 e-02 | 2.6042 e-02 | 2.6225 e-02 | 2.6417 e-02 | 2.5301 e-02 | 2.5314 e-02 |
| 1 | 0.01 25 | 0.01 25 | 25 | 500 | 2.5441 e-02 | 1.1042 e-01 | 2.3778 e-02 | 2.3905 e-02 | 2.6754 e-02 | 2.7528 e-02 | 2.8440 e-02 | 2.3580 e-02 | 2.3778 e-02 |
| 4 | 0.1 | 0.025 | 25 | 250 | 2.8505 e-02 | 1.0647 e-01 | 2.1632 e-02 | 4.5945 e-02 | 2.7752 e-02 | 2.9393 e-02 | 3.1494 e-02 | 2.0994 e-02 | 2.1634 e-02 |
| 4 | 0.2 | 0.05 | 25 | 125 | 4.3099 e-02 | 9.8129 e-02 | 1.6992 e-02 | 1.0276 e-01 | 2.9909 e-02 | 3.3434 e-02 | 3.8263 e-02 | 1.7677 e-02 | 1.7009 e-02 |
| 1.25 | 0.8 | 0.64 | 80 | 100 | 1.1817 e-01 | 2.3865 e+26 (Unst-able) | 6.1270 e+21 (Unstable) | 7.9059 e-01 | 2.7958 e-03 | 6.0336 e-02 | 1.0819 e-01 | 3.1393 e-02 | 4.7915 e-02 |
| 2.5 | 0.4 | 0.16 | 40 | 100 | 6.7932 e-02 | 6.9537 e-02 | 1.6090 e-03 | 3.0428 e-01 | 1.5658 e-02 | 3.0312 e-02 | 4.8274 e-02 | 3.1098 e-02 | 1.7711 e-03 |
| 5 | 0.2 | 0.04 | 20 | 100 | 5.1900 e-02 | 1.0188 e-01 | 2.5050 e-02 | 8.5124 e-02 | 4.1083 e-02 | 4.3673 e-02 | 4.7057 e-02 | 2.5019 e-02 | 2.5060 e-02 |

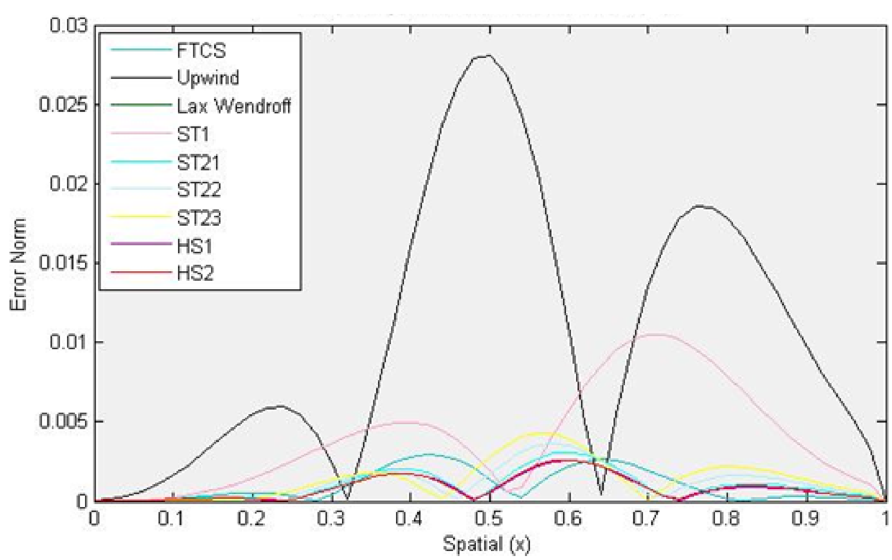



Fig. 1: Error Computation With Proposed Schemes And Conventional Methods

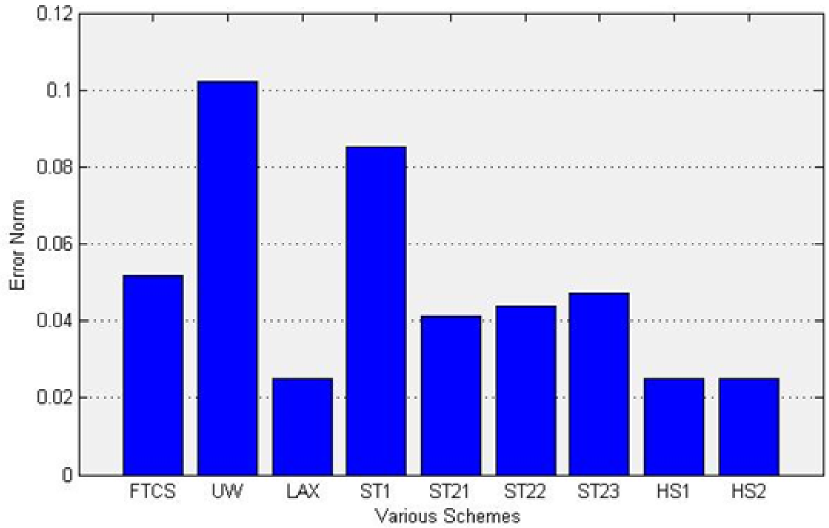



Fig. 2: Error Computation With Proposed Schemes And Conventional Methods

the corresponding (R), where M is the number of spatial (x) steps and N is the time steps.

It can be seen in Table. 1 with $c = 0.05$ and $s = 0.025$ the error of HS1 is $7.8735e - 03$ which is smaller than all existing methods such as FTCS, upwind, ST1, ST21, ST22 and ST23, however, the error of HS2 is $7.9619e - 03$ which is almost same that of Lax Wendroff. Similar performance can be seen with $R = 2$ for both New HS1 and HS2 methods.

In Table 1, it is also observed at the value of $R = 1.25$, the error grows rapidly in both Upwind and Lax Wendroff methods due to failure of stability requirement. However, the proposed methods HS1 and HS2 work well at the same value of R. HS2 has a smaller magnitude of error than all other methods. The performance of both proposed methods can also be seen in the remaining values of R in Table 1.

The flow of error at time $T = 1$ for all values of space $x$ are shown in Figure 1 with $c = 0.05$ and $s = 0.025$. Both new methods HS1 and HS2 graphs are below the others, confirming the smaller magnitude of errors than existing methods.

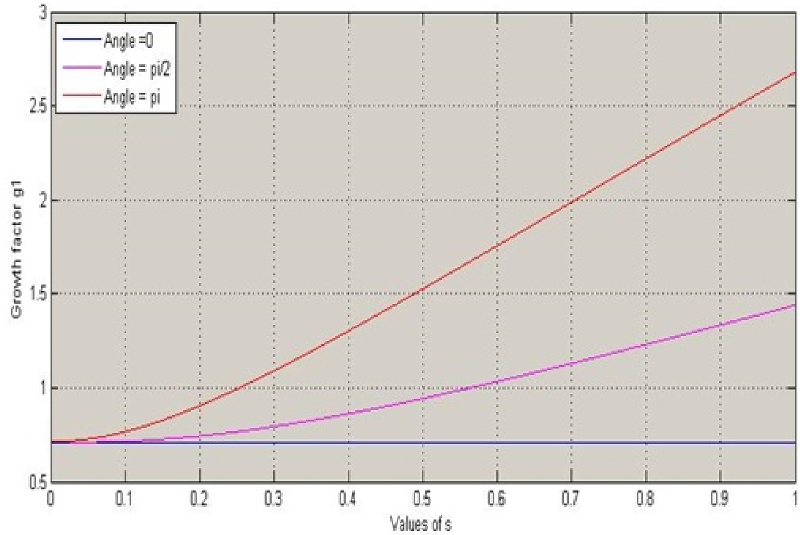
In Figure-2, the bar charts for various methods are shown at T=1 with $c = 0.2$ and $s = 0.04$, hence can be seen that the last two bars are smaller than other bars indicating HS1 and HS2 have smaller errors than Saulyev's HS1 and other methods.

In Figures 3 (a) and 3(b) the absolute values of growth factors —g— have been plotted from the Von Neumann Stability analysis of HS1. It is observed that for different values of angle $(0, \pi/2 and \pi)$ within the range $[0, \pi]$ the stability requirement $|g| \leq 1$ is satisfied by the method HS1. Hence, HS1 is conditionally stable within the range $0 \leq s \leq 0.2$, however, the value of $c$ may be restricted to the range $0 \leq c \leq 4$.

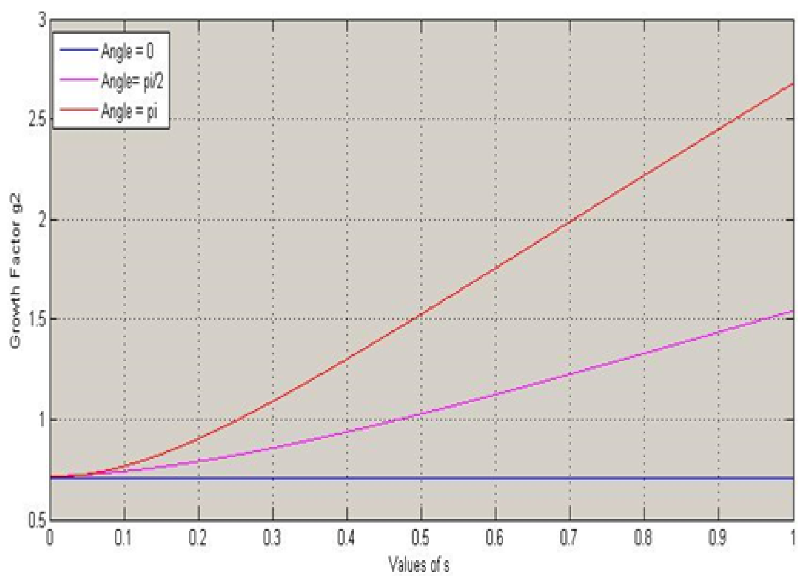
In Figure 4, the absolute values of the growth factor have been plotted from the Von Neumann stability analysis of HS2. It is observed that for values of angle with the range $[0, \pi]$ the stability requirement $|g| \leq 1$ is satisfied by method HS2. Hence, HS2 is conditional stable within $0 \leq s \leq 0.6$, however the value of c may be restricted to the range $0 \leq c \leq 0.42$.

## 7 Conclusion

The comparison of both proposed schemes has been carried out with other existing schemes in the literature. This shows satisfaction with the proposed schemes and produced smaller errors than the conventional methods. It is also observed that the schemes such as FTCS, Upwind, and Lax-Wendroff due to the stability requirement cannot be used where Saulyev's Type-II ST2-1, ST2-2, ST2-3 work well. However, the proposed schemes have better stability requirements such as HS2 is stable for all values of s with only restriction of c with the range $[0, 0.42]$. While HS1 has smaller errors in computation than other second-order conventional methods such as Lax-Wendroff and Saulyev's schemes. Both proposed methods are seen as robust in comparison to methods. Hence proposed schemes are explicit in nature and can be programmed-friendly. Hence schemes can be entertained in real-world problems for numerical solutions of 1-D advection diffusion Equations.




(a)




(b)

Fig. 3: (a) Von Neumann Stability Analysis of HS1, and (b) Von Neumann Stability Analysis of HS1

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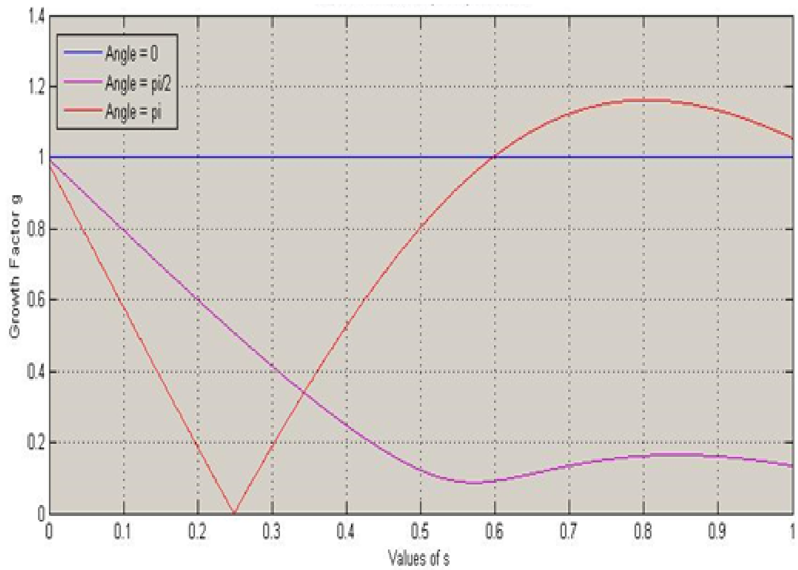


Fig. 4: Von Neumann Stability Analysis of HS2

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