Higher Order Runge-Kutta Method for Solving System of Ordinary Differential Equations Using MS Excel

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Abstract

The real world problems are often hard to understand when data is scattered or not logically organized in a way that is conducive for us. These problems are obtained from physical and scientific phenomena and are modelled in terms of differential equations. The differential equations, solved numerically, are often implemented on simulation and computational tools. This requires programming skills. Excel spreadsheet has been used frequently for statistical analysis, however, it is rarely used for computation in mathematics, instead of having great usability and usefulness. It also easily enables a non-programmer to perform programming-like tasks in a visual tabular pen and paper approach. Many physical phenomena are modelled in terms of odes and PDEs, therefore, solving these equations, by cooperative iterative methods, is very important. There are several iterative methods, among which Euler and Runge-Kutta are the most famous ones for solving the differential equations as well as system(s) of the differential equation(s). Euler's method has slow convergence; therefore, methods of a higher order of accuracy are often needed. Since the iterative methods manually with pen-paper are quite tedious and laboring because it involves numerous repetitive calculations. In this paper, the Excel spreadsheet is applied to solve the system of ordinary differential equations by applying the RK5 methods. It is demonstrated that the Excel spreadsheet method is faster and the solutions can be visualized by small or large alterations in the variables.

Keywords—Differential equation, Excel spreadsheet, Euler method, Runge Kutta method.

1 Introduction

In The scientific world is enriched daily with new knowledge, due to new technologies and continuous discoveries. Ordinary differential equations play vital role for mathematically modeling. Modeling complicated phenomena ends up in a differential equation or system of differential equations which are difficult to solve analytically. Numerical methods are resorted to obtain approximate solution; however, they require computations done by machine, thus required programming skills to implement numerical schemes and interpret result. The Excel spreadsheet has been lately used for calculus and numerical computations [1]. The mathematical functions explain the many basic terminologies of electrical concepts particularly those of voltage, current, resistance and graphical representations, elaborating processes based on the spreadsheet [2], [3], [4] operations. The main focus of this paper is to highlight the mathematical basis of electrical models that regulate the operation of spreadsheets in Microsoft Excel [3], [3], [5], [6], [7]. The spreadsheet application is among the most widely used computing tools in modern society because it is not possible to solve these equations analytically, so numerical solution is mandatory, there are numerous numerical methods available to solve odes [8], [9], [10], [11], among all the methods 5^th order Runge-Kutta method (RK5) [12], [13], [14], [15] is interesting.

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2 Problem Description: System of Ordinary Differential Equations

The general system of ordinary equations with initial conditions is given below

$$\frac{\mathrm{dX}}{\mathrm{d}t} = F(t, X, Y) \tag{1}$$

along with boundary conditions

$$\begin{aligned} X(t_0) &= X_0 \\ Y(t_0) &= Y_0 \end{aligned} \tag{2}$$

and $t_0 \leq t << t_n$. The solution domain is discretized such that; t_0, $t_1 = t_0 + \delta t$, $t_2 = t_0 + 2\delta t$, $t_3 = t_0 + 3\delta t$, $\ldots, t_n = t_0 + n\delta t$, where δt is the step size of t. For numerical experiments, we chose RL Circuit problem with two different parametric values.

2.1 RL Circuits

The series RL circuit of current I, resistance R, and inductance L, and electromotive force $E(t)$, as shown in Figure (1) is considered.

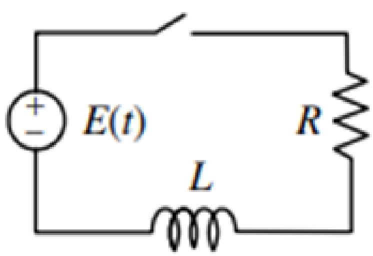

Fig. 1: A simple RL circuit

The governing equations defining relationship between the charge Q and the current I is given by;

$$\begin{aligned} \frac{dQ}{dt} &= I \\ \frac{d^2Q}{dt^2} &= \frac{dI}{dt} \end{aligned} \tag{3}$$

According to Kirchhoff's second law, the charge Q satisfies the second order ordinary differential equation

$$L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} = E\,(t)\,. \tag{4}$$

3 Butcher's 5^th Order Runge-Kutta Method

The numerical scheme of RK5 method is detailed out as

$$Q_{j+1} = Q_j + \frac{1}{90}\left[7F_1 + 32F_3 + 12F_4 + 32F_5 + 7F_6\right], \tag{5}$$

and

$$I_{j+1} = I_j + \frac{1}{90}[7G_1 + 32*G_3 + 12*G_4 + 32*G_5 + 7G_6] \tag{6}$$

where $t_{j+1} = t_j + \delta t$ and

$$\begin{aligned}
F_1 &= \delta t F\left(t_j, X_j, Y_j\right) \\
F_2 &= \delta t\ F\left(t_j + \frac{h}{4}, X_j + \frac{F_1}{2}, Y_j + \frac{G_1}{4}\right) \\
F_3 &= \delta t\ F(t_j + h/2, X_j + \frac{F_1}{8} + \frac{F_2}{8}, Y_j + \frac{G_1}{8}\ + \frac{G_2}{8} \\
F_4 &= \delta t\ F\left(t_j + \frac{h}{2}, X_j - \frac{F_2}{2} + \frac{F_3}{2}, Y_j - \frac{G_2}{2} + \frac{G_3}{2}\right) \\
F_5 &= \delta t\ F\left(t_j + \frac{3h}{4}, X_j + \frac{3F_1}{16} + \frac{F_4}{16}, Y_j + \frac{3G_1}{16} + \frac{9G_4}{16}\right) \\
F_6 &= \delta t\ F\left(t_j + h, X_j - \frac{3F_1}{7} + \frac{2F_2}{7} + \frac{12F_3}{7} - \frac{12F_4}{7} + \frac{8F_5}{7},\right. \\
&\qquad \left. Y_j - \frac{3G_1}{7} + \frac{2G_2}{7} + \frac{12G_3}{7} - \frac{12G_4}{7} + \frac{8G_5}{7}\right)
\end{aligned}$$

$$\begin{aligned}
G_1 &= \delta t\ G\left(t_j, X_j, Y_j\right) \\
G_2 &= \delta t\ G\left(t_j + \frac{h}{4}, X_j + \frac{F_1}{4}, Y_j + \frac{G_1}{4}\right) \\
G_3 &= \delta t G\left(t_j + \frac{h}{2}, X_j + F_{\frac{1}{8}} + \frac{F_2}{8}, Y_j + \frac{G_1}{8} + \frac{G_2}{8}\right) \\
G_4 &= \delta t\ G\left(t_j + \frac{h}{2}, X_j - \frac{F_2}{2} + \frac{F_3}{2}, Y_j - \frac{G_2}{2} + \frac{G_3}{2}\right) \\
G_5 &= \delta t\ G\left(t_j + \frac{3h}{4}, X_j + \frac{3F_1}{16} + \frac{F_4}{16}, Y_j + \frac{3G_1}{16} + \frac{9G_4}{16}\right) \\
G_6 &= \delta t\ G\left(t_j + h, X_j - (3F_1)/7 + (2F_2)/7 + 12\frac{F_3}{7}\right. \\
&\qquad -12F_4/7 + 8\frac{F_5}{7}), Y_j - \frac{3G_1}{7} + \frac{2G_2}{7} + 12\frac{G_3}{7} \\
&\qquad \left. -12\frac{G_4}{7} + 8\frac{G_5}{7}\right)
\end{aligned}$$

4 Numerical Experiments

Two specific problems are tested and results along with their implementation in spreadsheet is presented.

4.1 Problem 1

Let $R = 21$ Ohms, $L = 3$ Henrys, $E(t) = 126$ Volts and the starting setting time is zero. To determine I and Q in between $0 \ll t \ll 3$, with $\delta t = \frac{1}{4}$, using RK5 methods. An analytic solution for the Q, and I are given by;

$$Q = \frac{6}{7}\left(e^{-7t} - 1\right) + 6t \text{ and } I = 6\left(1 - e^{-7t}\right). \tag{7}$$

4.1.1 Implementation on Excel

Step 1: The 1st order differential equation, using Eq: (1), is given by

$$\frac{dQ}{dt} = I = F(t, Q, I)$$

$$\frac{dQ}{dt} = \frac{E(t) - R\,I}{5} = \frac{126 - 21\,I}{3} = 42 - 7I = G(t, Q, I).$$

Step 2: Writing RK5 formulae Eq: (3) and Eq: (4) in terms of Q and I is given below. Similarly, for RK5,

$$Q_{j+1} = X_j + \frac{1}{90}\left[7F_1 + 32F_3 + 12F_4 + 32F_5 + 7F_6\right] \tag{8}$$

$$I_{j+1} = I_j + \frac{1}{90}\left[7G_1 + 32G_3 + 12G_4 + 32G_5 + 7G_6\right] \tag{9}$$

where we choose $\delta t = 1/4$ and

$$F_1 = \frac{1}{4}I$$

$$F_2 = \frac{1}{4}\left(I + \frac{G_1}{2}\right)$$

$$F_3 = \frac{1}{4}\left(I + \frac{G_1}{8} + \frac{G_2}{8}\right)$$

$$F_4 = \frac{1}{4}\left(I - \frac{G_2}{2} + G_3\right)$$

$$F_5 = \frac{1}{4}\left(I + \frac{3G_1}{16} + \frac{9G_4}{16}\right)$$

$$F_6 = \frac{1}{4}\left(I - \frac{3G_1}{7} + \frac{2g_2}{7} + \frac{12g_3}{7} - \frac{12g_4}{7} + \frac{8g_5}{7}\right)$$

$$G_1 = \frac{1}{4}(42 - 7I)$$

$$G_2 = \frac{1}{4}\left[42 - 7I - \frac{7G_1}{4}\right]$$

$$G_3 = \frac{1}{4}\left[42 - 7I - \frac{7G_1}{8} - \frac{7G_2}{8}\right]$$

$$G_4 = \frac{1}{4}\left[42 - 7I + \frac{7}{2G_2} - 7G_3\right]$$

$$G_5 = \frac{1}{4}\left[42 - 7I - \frac{21G_1}{8} - \frac{63G_4}{8}\right]$$

$$G_6 = \frac{1}{4}\left[42 - 7I + 3G_1 - 2G_2 - 12G_3 + 12G_4 - 8G_5\right]$$

Step 3: The approximate solution is shown in Table 1

Discussion on results: It can be observed that for fixed T, and a fixed V of a series circuit is $T = \frac{L}{R} = \frac{3}{21} = 0.14286s$, and therefore T required for the current flowing in the RL circuit to approach its highest steady is about, $3T = 3(0.14286) = 0.429s$, as presented in Figure 3. Figure shows the current I at $V = 126$ volts for an RL series circuit. Since at the $I = 6$ amperes reach the steady-state solution as time tends to ∞. Here graphical analysis states that, the voltage V_L across the inductor L not only decays, but it decays exponentially from its starting point to 0, for a steady-state response. This is presented that voltage drop is;

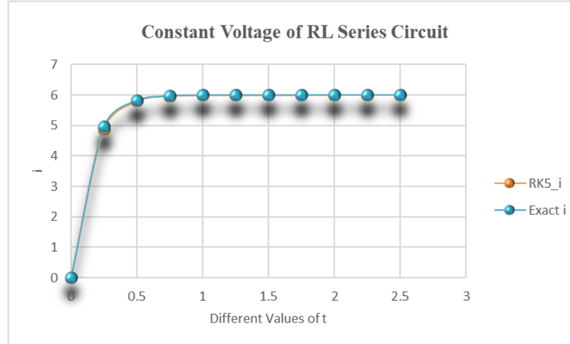

Fig. 2: A current of a fixed voltage series circuit RL

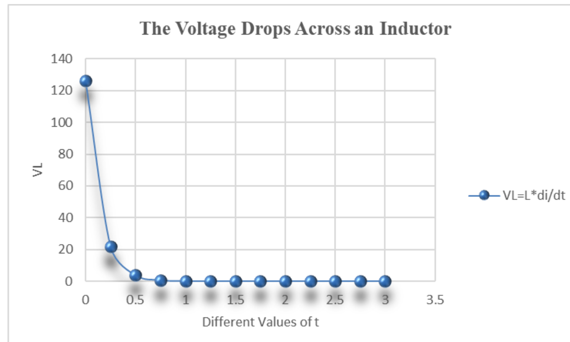

Fig. 3: A fixed voltage drops across the inductor for the LR Series Circuit.

$$V_L = L\frac{dI}{dt} = 3\left(42\,e^{-2t}\right) = 126e^{-7t}.$$

The approximate solution of ordinary differential equations by applying for the 5th order Runge Kutta method in Excel Spreadsheet is given by. The step wise pseudo algorithm is given in Algorithm 1. Next, we select the cells E5 to P5 from left to right and drag it from the right bottom corner of the cell P5 up to the right corner of P6, now again select the cells from B6:P6, then take the bottom right corner of P6 and double click will automatically fill the Excel Spread Sheet up to required value. It should be noted that the same technique applied on the same Spreadsheet with a different formula of RK5.

4.2 Problem 2 (The Sinusoidal Voltage RL Ciruit)

Considering $R = 21, L = 3, E(t) = 126\sin 2t$ and the starting point is when time is zero. To determine I and Q in between $0 \ll t \ll 3$, with $\delta t = 1/4$, using RK5

Algorithm 1 Step-wise pseudo algorithm

1: For the cell E5, type $A5 * D5 = f_1$.
2: For F5, type $=A5 * (42 - 7 * D5) = G_1$
3: Adjust the formula in cells:
$G5 : A5 * (D5 + F5/4) = F_2$
$H5 : A5 * (42 - 7 * (D5 + F5/4)) = G_2$
4: Formula routinely corrected as:
$I5 : A5 * (D5 + F5/8 + H5/8) = F_3$
$J5 : A5 * (42 - 7 * (D5 + F5/8 + H5/8)) = G_3$
5: For K5 and L5 type K5: $A5*(D5+H5/2+J5/2) = F_4$
and type $L5 : A5*(42-7*(D5+H5/8+J5/8)) = G_4$
6: For M5 and N5:
M5: $A5 * (D5 + 3/16F5 + 9/16L5) = F_5$
N5: $A5 * (42 - 7 * (D5 + 3/16F5 + 9/16L5)) = G_5$
7: For O5 type O5: $A5 * (D5 - 3/7F5 + 2/7H5 + 12/7J5 - 12/7L5 + 8/7N5) = F_6$
and P5: $A5 * (42 - 7 * (D5 - 3/7F5 + 2/7H5 + 12/7J5 - 12/7L5 + 8/7N5)) = G_6$
8: For C6 and D6 type the RK5 formulae:
$Q_(j+1) = Q_j + 1/90[7F_1 + 32 * F_3 + 12 * F_4 + 32 * F_5 + 7F_6]$
$I_(j+1) = I_j + 1/90[7G_1 + 32 * G_3 + 12 * G_4 + 32 * G_5 + 7G_6]$.

methods. An analytic solution for the Q, and I are given by;

$$Q(t) = 3 - \frac{21}{53}(7\cos 2t + 2\sin 2t) - \frac{12}{53}e^{-7t} \text{and}$$

$$I(t) = \frac{294}{53}\sin 2t - \frac{84}{53}\cos 2t + \frac{84}{53}e^{-7t}$$

4.2.1 Implantation on Excel

Step 1: The First order differential Equation

$$\frac{dQ}{dt} = I = F(t, Q, I)$$

$$\frac{dQ}{dt} = \frac{E(t) - RI}{3}$$

$$\frac{dQ}{dt} = \frac{126\sin 2t - 21\ I}{3} = 42\sin 2t - 7I = G(t, Q, I).$$

Step 2: Write the RK5 method Eq: (4), Eq: (5), in terms of Q, and I are given below:

$$Q_{j+1} = Q_j + \frac{1}{90}[7F_1 + 32F_3 + 12F_4 + 32F_5 + 7F_6],$$

and

$$I_{j+1} = I_j + \frac{1}{90}[7G_1 + 32G_3 + 12G_4 + 32G_5 + 7G_6]$$

where $\delta t = \frac{1}{4}$, and

$$F_1 = \frac{1}{4}I$$

$$F_2 = \frac{1}{4}\left(I + \frac{G_1}{2}\right)$$

$$F_3 = \frac{1}{4}\left(I + \frac{G_1}{8} + \frac{G_2}{8}\right)$$

$$F_4 = \frac{1}{4}\left(I - \frac{G_2}{2} + G_3\right)$$

$$F_5 = \frac{1}{4}\left(I + \frac{3G_1}{16} + \frac{9G_4}{16}\right)$$

$$F_6 = \frac{1}{4}\left[I - \frac{3G_1}{7} + \frac{2g_2}{7} + \frac{12g_3}{7} - \frac{12g_4}{7} + \frac{8g_5}{7}\right]$$

$$G_1 = \frac{1}{4}(42\sin 2t - 7I)$$

$$G_2 = \frac{1}{4}[42\sin\left(2t + \frac{1}{8}\right) - 7I - 7\frac{G_1}{2}]$$

$$G_3 = \frac{1}{4}[42\sin\left(2t + \frac{1}{8}\right) - 7I - 7\frac{G_1}{8} - 7\frac{G_2}{8}]$$

$$G_4 = \frac{1}{4}[42\sin\left(2t + \frac{1}{8}\right) - 7I + 7\frac{G_1}{2} - 7\frac{G_3}{2}]$$

$$G_5 = \frac{1}{4}\left[42\sin\left(2t + \frac{3}{16}\right) - 7I - 21\frac{G_1}{16} - 27\frac{G_4}{16}\right]$$

$$G_6 = \frac{1}{4}\left[42\sin\left(2t + \frac{1}{2}\right) - 7I + 3G_1 - 2G_2\right.$$
$$\left. - 12(G_3 + G_4) - 8G_5\right]$$

$$Q_{j+1} = Q_j + \frac{1}{90}[7 * F_1 + 32 * F_3 + 12 * F_4 + 32 * F_5 + 7 * F_6]$$

$$I_{j+1} = I_j + \frac{1}{90}[7 * G_1 + 32 * G_3 + 12 * G_4 + 32 * G_5 + 7 * G_6]$$

Step 3: The approximate solution with Excel Spreadsheet as shown in Table 2. Figure 5 presents the current of sinusoidal voltage for an RL circuit. An observation is made that, from the analytical solution, the current I have given in the Equation (1), the solution is steady-state and given by $I(t) = \frac{294}{53}\sin 2t - \frac{84}{53}\cos 2t$, as t approaches to ∞. This steady-state sinusoidal solution is presented in Figure 5, where the result is alike as the supplied voltage $E(t)$ as compared (Figure 5 and Figure 6).

An approximate solution that is plotted in Figure 4, resulting from the same steps of Excel Spreadsheet commands except for different functions.

5 Conclusion

An Excel spreadsheet presented to solve the system of ordinary differential equations by the RK5 method, it is found that the Excel Spreadsheet overcome flaps of other methods, and way spreadsheet suggested here is faster than any scientific calculator, and the solution

	Dt	t	f_1	f_2	f_3	f_4	f_5	f_6	App: RK5_q	Exact RK5_q	$\lvert Error \rvert$
1	0.25	0	0	0.656	0.513	0.990	0.995	1.738	0	0	0
2	0.25	0.25	1.220	1.342	1.315	1.405	1.406	1.544	0.803	0.792	0.011
3	0.25	0.5	1.448	1.471	1.466	1.482	1.482	1.508	2.173	2.169	0.004
4	0.25	0.75	1.490	1.494	1.494	1.496	1.497	1.501	3.648	3.647	0.001
5	0.25	1.0	1.50	1.499	1.499	1.499	1.499	1.500	5.144	5.143	0.0002
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	0.25	3	1.5	1.5	1.5	1.5	1.5	1.5	17.143	17.143	6.3E-09

TABLE 1: An approximate and an exact solution and an error of ODEs by RK5; $E(t) = 126volt$

	Dt	t	g_1	g_2	g_3	g_4	g_5	g_6	App: RK5_q	Exact RK5_q	$\lvert Error \rvert$
1	0.25	0	10.5	5.906	6.911	3.573	3.537	-1.665	0	0	0
2	0.25	0.25	1.963	1.104	1.292	0.668	0.661	-0.311	4.879	4.957	0.079
3	0.25	0.5	0.367	0.206	0.241	0.125	0.124	-0.068	5.790	5.819	0.029
4	0.25	0.75	0.069	0.039	0.045	0.023	0.023	-0.011	5.960	5.969	0.008
5	0.25	1.0	0.013	0.007	0.008	0.004	0.004	0.002	5.993	5.95	0.002
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	0.25	3	1.9E-8	1.1E-8	1.2E-8	1.E-9	1.4E-9	-3.1E-9	6	6	6.4E-9

TABLE 2: An approximate and an exact solution and an error of ODEs by RK5; $E(t) = 126volt$

	Dt	t	f_1	f_2	f_3	f_4	f_5	f_6	App: RK5_q	Exact RK5_q	$\lvert Error \rvert$
1	0.25	0	0	0	0.041	0.092	0.275	0.244	0	0	0
2	0.25	0.25	0.1435	0.5349	0.5382	0.686	0.831	0.967	0.144	0.147	0.003
3	0.25	0.5	0.8227	1.095	1.087	1.206	1.277	1.409	0.827	0.828	0.0005
4	0.25	0.75	2.013	1.417	1.404	1.449	1.428	1.497	2.013	2.012	0.0011
5	0.25	1.0	3.4353	1.397	1.383	1.341	1.233	1.218	3.435	3.433	0.0018
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	0.25	5.0	5.759	-0.595	-0.59	0.756	-0.89	-1.06	5.758	5.758	0.0003

TABLE 3: An approximate and an exact solution and an error of ODEs by RK5; $E(t) = 126 \sin 2t$volt

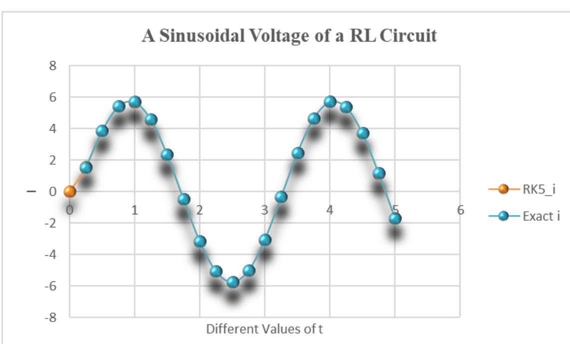

Fig. 4: A fixed voltage drops across the inductor for the LR Series Circuit.

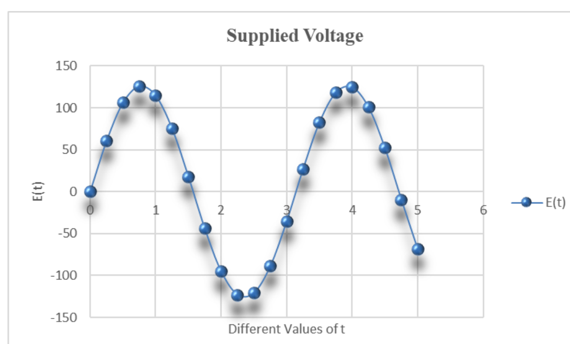

Fig. 5: A fixed voltage drops across the inductor for the LR Series Circuit.

obtained is notably more literal, which can be visualized by small or large alterations in the variables if needed. Besides, this method presented here can be extended to evaluate systems of odes up to n equations; it does provide a step-by-step environment for generous and knowledge of numerical methods.

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