Computationally Efficient Three-Step Derivative-Free Iterative Scheme for Nonlinear Algebraic and Transcendental Equations

Ramesh Meghwar[1], Zubair Ahmed Kalhoro[1], Sanaullah Jamali[2]

[1]Institute of Mathematics and Computer Science, University of Sindh, Jamshoro, Pakistan
[2]University of Sindh, Laar Campus, Badin, Pakistan
[*]Corresponding author: sanaullah.jamali@usindh.edu.pk

Abstract

The solution of nonlinear algebraic and transcendental equations is fundamental in various fields of science and engineering, with applications from mathematical modeling to biological modeling, including population growth, blood rheology, and neurophysiology. Traditional methods, such as Newton's method, often rely on derivatives and face challenges like slow convergence, high computational cost, and failure in cases where derivatives are difficult or impossible to compute. This research presents an efficient derivative-free iterative scheme constructed by Obadah Said Solaiman's method and Newton method in which derivatives are approximated by using forward difference and Pade approximation. The proposed scheme is designed to achieve rapid convergence while minimizing computational cost. Numerical experiments are conducted to compare the performance of the proposed scheme against existing methods, highlighting its superior efficiency and reliability in various test problems. The results show that the developed scheme is useful for solving nonlinear equations, particularly in scenarios where derivative computation is infeasible or computationally expensive.

Keywords—Nonlinear equations, Iterative scheme, Derivative-free, Pade approximation, Order of convergence, Efficiency index reluctance motor

1 Introduction

Numerical analysis is one of the branches of mathematics that provides multiple schemes for solving mathematical problems [1]. Numerical analysis has a prominent position in engineering and physical science [2], and basically, it is all about algorithms that use numerical approximations to overcome the difficulties of mathematical analysis. The goal of the field of numerical analysis is very clear: firstly arrange and examine the way of working then to give approximate and accurate solutions to difficult problems [3].

Numerous studies have been conducted to provide simple, affordable, and derivative-free approaches to address the nonlinear problems that emerge in day-to-day living. Many studies have developed bracketing methods utilizing different techniques; some of them are included in [4–6]. Although bracketing methods usually converge to the root, their convergence is relatively sluggish, and they need two initial guesses, dependency of initial guesses, and stability issues [7].

Additionally, there are several studies that have developed several open multistep methods using various techniques, i.e, Lagrange interpolation technique [2], Hermite interpolation technique [8], weight functions [9]. The efficiency index ($p^(1/k)$, where p is the order of convergence and k is the number of functional evaluations) has been boosted [10]. And, also, there are multiple scholars working and developing derivative free open multistep methods [11–13]. To validate the proposed approach, we compare it to numerous seventh-order methods that exist in the literature. M1 [14] evaluates four functions and one first derivative every iteration, with an efficiency index of 1.4757. M2 [15] employs two functions, two first, and second two derivative assessments, reducing efficiency to 1.3830. M3 [16], M4 [17], M5 [18] have the same structure, which requires three functions and two first derivative evaluations while retaining 1.4757 efficiency. And, M6 [19] needs three functions and three first derivative evaluations, which reduces efficiency to 1.3831.

ISSN: 2523-0379 (Online), ISSN: 1605-8607 (Print)
DOI: https://doi.org/10.52584/QRJ.2301.04

2 Derivation of Proposed Method

This section will first outline the concepts of the iterative approach put forth by Obadah Said Solaiman, which is based on Halley's method and the Taylor series expansion. After that, we will present a new, effective derivative-free iterative scheme for solving nonlinear equations, which typically employ iterative techniques like Newton's method. The most popular iterative technique is like Newton's approach for calculating the simple root of a nonlinear equation $f(\beta) = 0$, which employs two function evaluations every iteration: one function and one first-order derivative. Now, we will consider the following two-step method: Obadah Said Solaiman, algorithm.

$$\gamma_n = \beta_n - \frac{f(\beta_n)}{f'(\beta_n)},$$

$$\beta_{n+1} = \gamma_n - \frac{f(\gamma_n)}{f'(\gamma_n)} - \frac{2\,f(\gamma_n)^2 f'(\gamma_n) f''(\gamma_n)}{4f'(\gamma_n)^4 - 4f(\gamma_n)\,f'(\gamma_n)^2 f''(\gamma_n) + f(\gamma_n)^2 f''(\gamma_n)^2}. \tag{A}$$

It is referred to as modified Halley's method and has a sixth order of convergence. The method consists of two evaluations of the function, two evaluations of the first derivative, and one evaluation of the second derivative per iteration, which has an efficiency index of 1.4309. The aim is to modify the above two-step method to a three-step method in order to reduce the derivation evaluation using Pade approximation and make it a derivative free method by Steffensen's rule. We employ Newton's technique in the third step of (A) to enhance the convergence rate. The new multistep technique that we acquired is as follows:

$$\gamma_n = \beta_n - \frac{f(\beta_n)}{f'(\beta_n)},$$

$$\delta_n = \gamma_n - \frac{f(\gamma_n)}{f'(\gamma_n)} - \frac{2f(\gamma_n)^2 f'(\gamma_n) f''(\gamma_n)}{4f'(\gamma_n)^4 - 4f(\gamma_n)f'(\gamma_n)^2 f''(\gamma_n) + f(\gamma_n)^2 f''(\gamma_n)^2},$$

$$\beta_{n+1} = \delta_n - \frac{f(\delta_n)}{f'(\delta_n)}. \tag{B}$$

This method consists of three evaluations of the function, three evaluations of the first derivative, and one evaluation of the second derivative per iteration. So, the efficiency index of this scheme is 1.3204. To derive a scheme with a higher efficiency index, we approximate: $f^{'}(\beta_n)$ Using forward-difference, $f^{'}(\gamma_n)$ Using forward-difference, $f^{''}(\gamma_n)$ Using Taylor Series expansion, $f^{'}(\delta_n)$ Using a pade-approximation. Approximation to the $f^{'}(\beta_n)$ by the forward-difference approximation

$$f'(\beta_n) \approx \frac{f(\beta_n + f(\beta_n)) - f(\beta_n)}{\beta_n + f(\beta_n) - \beta_n} \approx \frac{f(\beta_n + f(\beta_n)) - f(\beta_n)}{f(\beta_n)} \approx f[\beta_n,\ \beta_n + f(\beta_n)]. \tag{i}$$

Approximation to the $f^{'}(\gamma_n)$ by the forward-difference approximation

$$f'(\gamma_n) \approx \frac{\dfrac{f(\gamma_n) - f(\beta_n)}{\gamma_n - \beta_n} \quad \dfrac{f(\gamma_n) - f(\beta_n + f(\beta_n))}{\gamma_n - \beta_n - f(\beta_n)}}{\dfrac{f(\beta_n) - f(\beta_n + f(\beta_n))}{-f(\beta_n)}} \approx \frac{\dfrac{f(\gamma_n) - f(\beta_n)}{\gamma_n - \beta_n} \quad \dfrac{f(\gamma_n) - f(\beta_n + f(\beta_n))}{\gamma_n - \beta_n - f(\beta_n)}}{\dfrac{f(\beta_n + f(\beta_n)) - f(\beta_n)}{f(\beta_n)}} \approx \frac{f[\gamma_n, \beta_n]\; f[\gamma_n, \beta_n + f(\beta_n)]}{f[\beta_n, \beta_n + f(\beta_n)]}. \tag{ii}$$

To approximate $f^{''}(\gamma_n)$ we use the Taylor Series expansion of the function $f(\beta_n)$ around $\beta = \gamma_n$

$$f(\beta_n) \approx f(\gamma_n) + f'(\gamma_n)(\beta_n - \gamma_n) + \frac{f''(\gamma_n)(\beta_n - \gamma_n)^2}{2}$$

$$\approx \frac{f(\beta_n) - f(\gamma_n)}{\beta_n - \gamma_n} - \frac{f''(\gamma_n)(\beta_n - \gamma_n)}{2}$$

$$\approx \frac{2}{\beta_n - \gamma_n}\left[\frac{f(\beta_n) - f(\gamma_n)}{\beta_n - \gamma_n} - f'(\gamma_n)\right]$$

$$f''(\gamma_n) \approx \frac{2}{\beta_n - \gamma_n}\left[f[\beta_n, \gamma_n] - f'(\gamma_n)\right]. \tag{iii}$$

$$m(t) = \frac{a_0 + a_1(t - \delta_n) + a_2(t - \delta_n)^2 + \cdots}{1 + b_1(t - \delta_n) + b_2(t - \delta_n)^2 + \cdots}$$

Pade approximation of first order

$$m(t) = \frac{a_0 + a_1(t - \delta_n)}{1 + b_1(t - \delta_n)} \tag{1}$$

Where a_0, a_1, b_1 are real parameters to be determined satisfying the following conditions

$$f(\beta_n) = m(\beta_n)$$
$$f(\gamma_n) = m(\gamma_n)$$
$$f(\delta_n) = m(\delta_n)$$

Using these equations in equation (1), we get

$$f(\beta_n) = \frac{a_0 + a_1(\beta_n - \delta_n)}{1 + b_1(\beta_n - \delta_n)} \quad (2)$$

$$f(\gamma_n) = \frac{a_0 + a_1(\gamma_n - \delta_n)}{1 + b_1(\gamma_n - \delta_n)} \quad (3)$$

$$f(\delta_n) = a_0 \quad (4)$$

Using equation (4) in equation (2), we get

$$f(\beta_n) = \frac{f(\delta_n) + a_1(\beta_n - \delta_n)}{1 + b_1(\beta_n - \delta_n)}$$
$$(1 + b_1(\beta_n - \delta_n))f(\beta_n) = f(\delta_n) + a_1(\beta_n - \delta_n)$$
$$a_1(\beta_n - \delta_n) = f(\beta_n) - f(\delta_n) + b_1(\beta_n - \delta_n)f(\beta_n)$$
$$a_1 = \frac{f(\beta_n) - f(\delta_n)}{\beta_n - \delta_n} + b_1 f(\beta_n)$$
$$a_1 = f[\delta_n, \beta_n] + b_1 f(\beta_n) \quad (5)$$

Using equation (4) in equation (3), we get

$$f(\gamma_n) = \frac{f(\delta_n) + a_1(\gamma_n - \delta_n)}{1 + b_1(\gamma_n - \delta_n)}$$
$$f(\gamma_n) + b_1(\gamma_n - \delta_n)f(\gamma_n) = f(\delta_n) + a_1(\gamma_n - \delta_n)$$
$$a_1(\gamma_n - \delta_n) = f(\gamma_n) - f(\delta_n) + b_1(\gamma_n - \delta_n)f(\gamma_n)$$
$$a_1 = \frac{f(\gamma_n) - f(\delta_n)}{\gamma_n - \delta_n} + b_1 f(\gamma_n)$$
$$a_1 = f[\delta_n, \gamma_n] + b_1 f(\gamma_n) \quad (6)$$

From equations (5) and (6), we have

$$f[\delta_n, \beta_n] + b_1 f(\beta_n) = f[\delta_n, \gamma_n] + b_1 f(\gamma_n)$$
$$f[\delta_n, \beta_n] - f[\delta_n, \gamma_n] = b_1 f(\gamma_n) - b_1 f(\beta_n)$$
$$b_1 [f(\gamma_n) - f(\beta_n)] = f[\beta_n, \gamma_n] - f[\gamma_n, \delta_n]$$

$$b_1 = \frac{f[\delta_n, \beta_n] - f[\delta_n, \gamma_n]}{f(\gamma_n) - f(\beta_n)}$$
$$b_1 = -\frac{f[\gamma_n, \delta_n] - f[\delta_n, \beta_n]}{f(\gamma_n) - f(\beta_n)}$$
$$b_1 = -\frac{\frac{f[\gamma_n,\delta_n] - f[\delta_n,\beta_n]}{\gamma_n - \beta_n}}{\frac{f(\gamma_n) - f(\beta_n)}{\gamma_n - \beta_n}}$$
$$b_1 = -\frac{f[\beta_n, \gamma_n, \delta_n]}{f[\beta_m, \gamma_n]}$$

Using the value of b_1 into equation (6) we get

$$a_1 = f[\delta_n, \beta_n] - \frac{f[\beta_n, \gamma_n, \delta_n]}{f[\beta_m, \gamma_n]} f(\beta_n)$$

Now the derivative of equation (1)

$$m'(t) = \frac{a_1(1 + b_1(t - \delta_n)) - (a_0 + a_1(t - \delta_n))b_1}{(1 + b_1(t - \delta_n))^2}$$
$$m'(t) = \frac{a_1 + a_1 b_1 t - a_1 b_1 \delta_n - a_0 b_1 - a_1 b_1 t + a_1 b_1 \delta_n}{(1 + b_1(t - \delta_n))^2}$$
$$m'(t) = \frac{a_1 - a_0 b_1}{(1 + b_1(t - \delta_n))^2}$$
$$\text{As } f'(\delta_n) = m'(\delta_n)$$
$$f'(\delta_n) = a_1 - a_0 b_1$$
$$f'(\delta_n) = f[\delta_n, \beta_n] - \frac{f[\beta_n, \gamma_n, \delta_n]}{f[\beta_n, \gamma_n]} f(\beta_n) + f(\delta_n) \frac{f[\beta_n, \gamma_n, \delta_n]}{f[\beta_n, \gamma_n]}$$
$$f'(\delta_n) = f[\delta_n, \beta_n] + \frac{f[\beta_n, \gamma_n, \delta_n]}{f[\beta_n, \gamma_n]} (f(\delta_n) - f(\beta_n))$$

$$\left.\begin{aligned}
\gamma_n &= \beta_n - \frac{f(\beta_n)}{f[\beta_n, \beta_n + f(\beta_n)]} \\
\delta_n &= \gamma_n - \frac{f(\gamma_n)}{p_n} - \frac{2f^2(\gamma_n)p_n q_n}{4p_n^4 - 4f(\gamma_n)p_n^2 q_n + f^2(\gamma_n)q_n^2} \\
\beta_{n+1} &= \delta_n - \frac{f(\delta_n)}{R_n} \\
\text{where } p_n &= \frac{f[\gamma_n, \beta_n] f[\gamma_n, \beta_n + f(\beta_n)]}{f[\beta_n, \beta_n + f(\beta_n)]}, \\
q_n &= \frac{2}{\beta_n - \gamma_n} \left[f[\beta_n, \gamma_n] - p_n \right], \\
R_n &= \frac{f[\gamma_n, \delta_n] f[\beta_n, \delta_n]}{f[\beta_n, \gamma_n]}
\end{aligned}\right\} \quad \text{(V)}$$

This is a three-step seventh-order derivatives-free iterative scheme for finding solutions of nonlinear equations. The presented scheme requires only four function evaluations and does not require any derivatives, and the efficiency index of this scheme is 1.6266.

3 Convergence Analysis

$$\gamma_n = \beta_n - \frac{f(\beta_n)}{f[\beta_n, \beta_n + f(\beta_n)]}$$
$$\delta_n = \gamma_n - \frac{f(\gamma_n)}{p_n} - \frac{2f^2(\gamma_n)p_n q_n}{4p_n^4 - 4f(\gamma_n)p_n^2 q_n + f^2(\gamma_n)q_n^2}$$
$$\beta_{n+1} = \delta_n - \frac{f(\delta_n)}{R_n}$$
$$\text{where } p_n = \frac{f[\gamma_n, \beta_n] f[\gamma_n, \beta_n + f(\beta_n)]}{f[\beta_n, \beta_n + f(\beta_n)]},$$
$$q_n = \frac{2}{\beta_n - \gamma_n} [f[\beta_n, \gamma_n] - p_n],$$
$$R_n = \frac{f[\gamma_n, \delta_n] f[\beta_n, \delta_n]}{f[\beta_n, \gamma_n]}$$

Theorem I: Let $\alpha \in D$ be a simple root of a sufficiently differentiable function $f\colon D \subset R \to R$, where D is an open interval with β_0 as an initial estimate of α. With just four function evaluations, no derivative evaluations are required. The approach outlined in equation (8) has seventh-order accuracy under these circumstances.

Proof:

Taylor series extension allows us to express the function $f(\beta_n)$ as follows:

$$f\left(\beta_n\right)=\sum_{m=0}^{\infty}\frac{f^m(\alpha)}{m!}\left(\beta_n-\alpha\right)^m=f(\alpha)+f'(\alpha)\left(\beta_n-\alpha\right)+\frac{f''(\alpha)}{2!}\left(\beta_n-\alpha\right)^2+\frac{f'''(\alpha)}{3!}\left(\beta_n-\alpha\right)^3+\cdots$$

For simplicity, we assume that $C_k=\left(\frac{1}{k!}\right)\frac{f^k(\alpha)}{f'(\alpha)}, k=2,3,\ldots.$ and assume that $e_n=\beta_n-\alpha$. Thus, we have For step one:

$$f\left(\beta_n\right)=f'(a)\left(e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+\cdots+O\left(e_n^8\right)\right)$$

$$\begin{aligned}f\left(\beta_n+f\left(\beta_n\right)\right)=f'(a)\Big(&\left(1+f'(a)\right)e_n+c_2\left(1+3f'(a)+f'^2(a)\right)e_n^2\\&+\left(c_3f'(a)+2c_2^2f'(a)\left(1+f'(a)\right)+c_3\left(1+f'(a)\right)^3\right)e_n^3+\cdots+O\left(e_n^8\right)\Big)\end{aligned}$$

9

$$f\left[\beta_n,\beta_n+f\left(\beta_n\right)\right]=\frac{f\left(\beta_n+f\left(\beta_n\right)\right)-f\left(\beta_n\right)}{f\left(\beta_n\right)}=f'(a)\left(1+c_2\left(2+f'(a)\right)e_n+\left(c_2^2f'(a)+c_3\left(3+3f'(a)+f'(a)^2\right)\right)e_n^2\right.$$
$$\left.+\left(2+f'(a)\right)\left(2c_2c_3f'(a)+c_4\left(2+2f'(a)+f'(a)^2\right)\right)e_n^3+\cdots+O\left(e_n^8\right)\right)$$

$$\frac{f\left(\beta_n\right)}{f\left[\beta_n,\beta_n+f\left(\beta_n\right)\right]}=e_n-c_2\left(1+f'(a)\right)e_n^2+\left(c_2^2\left(2+2f'(a)+f'(a)^2\right)-c_3\left(2+3f'(a)+f'(a)^2\right)\right)e_n^3+\cdots+O\left(e_n^8\right)$$

$$\gamma_n=\beta_n-\frac{f\left(\beta_n\right)}{f\left[\beta_n,\beta_n+f\left(\beta_n\right)\right]}=c_2\left(1+f'(a)\right)e_n^2+\left(-c_2^2\left(2+2f'(a)+f'(a)^2\right)+c_3\left(2+3f'(a)+f'(a)^2\right)\right)e_n^3$$
$$+\cdots+O\left(e_n^8\right)$$

$$f\left(\gamma_n\right)=f'(a)\Big(c_2\left(1+f'(a)\right)e_n^2+\left(-c_2^2\left(2+2f'(a)+f'(a)^2\right)+c_3\left(2+3f'(a)+f'(a)^2\right)\right)e_n^3+\cdots+O\left(e_n^8\right)\Big)$$

9

$$p_n=\frac{f\left[\gamma_n,\beta_n\right]\,f\left[\gamma_n,\beta_n+f\left(\beta_n\right)\right]}{f\left[\beta_n,\beta_n+f\left(\beta_n\right)\right]}=f'(a)\Big(1+\left(3c_2^2-c_3\right)\left(1+f'(a)\right)e_n^2$$
$$-\left(c_4\left(2+3f'(a)+f'(a)^2\right)+c_2^3\left(8+9f'(a)+4f'(a)^2\right)-c_2c_3\left(10+14f'(a)+5f'(a)^2\right)\right)e_n^3+\cdots+O\left(e_n^8\right)\Big)$$

$$q_n=\frac{2}{\beta_n-\gamma_n}\left[f\left[\beta_n,\gamma_n\right]-p_n\right]=f'(a)\Big(2c_2-2\left(c_2^2\left(1+f'(a)\right)-c_3\left(2+f'(a)\right)\right)e_n$$
$$+2\left(c_2^3\left(3+3f'(a)+f'(a)^2\right)+c_4\left(3+3f'(a)+f'(a)^2\right)-c_2c_3\left(3+4f'(a)+2f'(a)^2\right)\right)e_n^2+\cdots+O\left(e_n^8\right)\Big)$$

$$\delta_n = \gamma_n - \frac{f(\gamma_n)}{p_n} - \frac{2f^2(\gamma_n)p_n q_n}{4p_n^4 - 4f(\gamma_n)p_n^2 q_n + f^2(\gamma_n)q_n^2} = c_2(c_2^2 - c_3)(1+f'(a))^2 e_n^4$$
$$- (1+f'(a))\Big(c_3^2(2+3f'(a)+f'(a)^2) + c_2c_4(2+3f'(a)+f'(a)^2) - 2c_2^2c_3(4+5f'(a)+2f'(a)^2)$$
$$+ c_2^4(5+5f'(a)+2f'(a)^2)\Big)e_n^5 + \cdots + O(e_n^8)$$

$$f(\delta_n) = f'(a)\Big(c_2(c_2^2-c_3)(1+f'(a))^2 e_n^4$$
$$- (1+f'(a))(c_3^2(2+3f'(a)+f'(a)^2) + c_2c_4(2+3f'(a)+f'(a)^2) - 2c_2^2c_3(4+5f'(a)+2f'(a)^2)$$
$$+ c_2^4(5+5f'(a)+2f'(a)^2))e_n^5 + \cdots + O(e_n^8)\Big)$$

$$P_n = \frac{f[\gamma_n,\delta_n]\, f[\beta_n,\delta_n]}{f[\beta_n,\gamma_n]} = f'(a)\Big(1 + c_2(c_2^2-c_3)(1+f'(a))e_n^3$$
$$+ \big(-c_2c_4(1+f'(a)) + c_2^2c_3(4+3f'(a)) + c_2^4(-1+f'(a)+f'(a)^2) - c_3^2(2+3f'(a)+f'(a)^2)\big)e_n^4$$
$$+ \cdots + O(e_n^8)\Big)$$

$$\frac{f(\delta_n)}{P_n} = c_2(c_2^2-c_3)(1+f'(a))^2 e_n^4 - (1+f'(a))\Big(c_3^2(2+3f'(a)+f'(a)^2) + c_2c_4(2+3f'(a)+f'(a)^2)$$
$$- 2c_2^2c_3(4+5f'(a)+2f'(a)^2) + c_2^4(5+5f'(a)+2f'(a)^2)\Big)e_n^5 + \cdots + O(e_n^8)$$

$$\beta_{n+1} = \delta_n - c_2(c_2^2-c_3)(1+f'(a))^2 e_n^4$$
$$- (1+f'(a))\Big(c_3^2(2+3f'(a)+f'(a)^2) + c_2c_4(2+3f'(a)+f'(a)^2)$$
$$- 2c_2^2c_3(4+5f'(a)+2f'(a)^2) + c_2^4(5+5f'(a)+2f'(a)^2)\Big)e_n^5$$
$$+ \cdots + O(e_n^8)$$

Finally, subtracting a from both sides, we get

$$\mathrm{e}_{n+1} = (c_2^3 - c_2c_3)^2(1+f'(a))^3 e_n^7 + O(e_n^8)$$

TABLE 1: Comparison table of the number of function evaluations used per iteration in each method

Methods	Number of evaluations per iteration
M1	$4n + n^2$
M2	$2n + 2n^2 + 2n^3$
M3	$3n + 2n^2$
M4	$3n + 2n^2$
M5	$3n + 2n^2$
M6	$3n + 4n^2$
PS	$4n$

4 Results and Analysis

All the problems below were solved using Maple 2022 software and origin 2021 software were used for graphs on a laptop with the specifications: Intel(R) Core (TM) i3-4010U CPU @ 1.70GHz, 1.70 GHz, and 8.00 GB RAM.

Problem 1: The depth of embedment γ in a sheet-pile wall [21, 22]. $f(\beta) = \frac{1}{4.62}\left(\beta^3 + 2.87\beta^2 - 10.28\right) - \beta$ Initial guess $\beta_0 = 1.0$

Problem 2: Consider the beam scheming problem [23]. $f(\beta) = \frac{1}{4.62}\left(\beta^3 + 2.87\beta^2 - 4.62\beta - 10.28\right)$ Initial guess $\beta_0 = 0.99$

Problem 3: The Plank's radiance law problem appears in [24]. $f(\beta) = \frac{8\pi hc\beta^{-5}}{e^{\frac{hc}{\beta TK^{-1}}}}$

This equation computes the density of energy in an isothermal blackbody. The equation is rewritten as: $f(\beta) = 1 - 0.2\beta - e^{-\beta}$, with an initial guess $\beta_0 = 0.35$.

Problem 4: Blood rheology model [24, 25]. $f(\beta) = \frac{1}{441}\beta^8 - \frac{8}{63}\beta^5 - 0.05714285714\beta^4 + \frac{16}{9}\beta^2 - 3.624489796\beta + 0.3$, with the initial guess $\beta_0 = 1.5$.

5 Discussion

This article has chosen a variety of application problems from different fields and tested them in order to validate the results of the proposed technique with the existing methods. The proposed approach performs better than the existing approaches in every test problem when using the fewest number of function evaluations possible without evaluating derivatives.

6 Conclusion

This research article presents a three-step seventh-order derivatives-free iterative scheme that utilizes Pade Approximation. Remarkably, this scheme requires only four function evaluations per iteration and exhibits superior performance as compared to existing approaches. The results, displayed in Tables 1-2 and Figures 1, clearly demonstrate that the proposed scheme achieves faster convergence across all test problems, requiring fewer iterations and function evaluations. In comparison, counterparts M1, M2, M3, M4, M5, and M6 have efficiency indices 1.4757, 1.3830, 1.4757, 1.4757, 1.4757, and 1.3204, respectively, and the proposed scheme has an efficiency index of 1.6266. This points out that the proposed scheme not only outperforms but also demonstrates a higher efficiency index compared to its counterparts' methods.

7 Acknowledgement

I am grateful to my M.Phil. supervisor and co-supervisors for their contributions to this study; without their direction and assistance, it would not have been practicable

References

[1] G. Gulshan, H. Budak, R. Hussain, and A. Sadiq, "Generalization of the bisection method and its applications in nonlinear equations," Advances in Continuous and Discrete Models, vol. 2023, no. 1, 2023, doi: 10.1186/s13662-023-03765-5.

[2] F. A. Lakho, Z. A. Kalhoro, S. Jamali, A. W. Shaikh, and J. Guan, "A three steps seventh order iterative method for solution nonlinear equation using Lagrange interpolation technique," VFAST Transactions on Mathematics, vol. 12, no. 1, pp. 46–59, 2024, doi: 10.21015/vtm.v12i1.1712.

[3] F. A. Shah, M. A. Noor, and K. I. Noor, "Some iterative schemes for obtaining approximate solution of nonlinear equations," UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, vol. 78, no. 1, pp. 59–70, 2016.

[4] S. Jamali, Z. A. Kalhoro, A. W. Shaikh, and M. S. Chandio, "An iterative, bracketing & derivative-free method for numerical solution of non-linear equations using Stirling interpolation technique," Journal of Mechanics of Continua and Mathematical Sciences, vol. 16, no. 6, pp. 13–27, Jun. 2021, doi: 10.26782/jmcms.2021.06.00002.

[5] B. M. Faraj, S. K. Rahman, D. A. Mohammed, B. M. Hussein, B. A. Salam, and K. R. Mohammed, "An improved bracketing method for numerical solution of nonlinear equations based on Ridders method," Matrix Science Mathematics, vol. 6, no. 2, pp. 30–33, 2022, doi: 10.26480/msmk.02.2022.30.33.

[6] M. I. Soomro, Z. A. Kalhoro, A. W. Shaikh, S. Jamali, and O. Ali, "Modified bracketing iterative method for solving nonlinear equations," VFAST Transactions on Mathematics, vol. 12, no. 1, pp. 105–120, 2024, doi: 10.21015/vtm.v12i1.1761.

[7] S. Intep, "A review of bracketing methods for finding zeros of nonlinear functions," Applied Mathematical Sciences, vol. 12, no. 3, pp. 137–146, 2018, doi: 10.12988/ams.2018.811.

[8] S. Jamali, Z. A. Kalhoro, A. W. Shaikh, and M. S. Chnadio, "Solution of chemical engineering models and their dynamics using a new three-step derivative free optimal method," Journal of Hunan University Natural Sciences, vol. 50, no. 1, pp. 236–245, Feb. 2023, doi: 10.55463/issn.1674-2974.50.1.24.

[9] L. Fang and L. Pang, "Seventh-order convergent iterative methods for solving nonlinear equations," International Journal of Applied Sciences and Mathematics, vol. 3, no. 6, pp. 195–197, 2016.

TABLE 2: Comparison table of absolute functional value up to 4th iteration of Problem 1 to 4 by the proposed method to counterpart M1 to M6

Problems	M1	M2	M3	M4	M5	M6	PM
Problem 1 $\beta_0 = 1.0$	2.550×10^{0} 1.954×10^{-4} 5.762×10^{-32} 1.118×10^{-224}	4.220×10^{1} 9.766×10^{-2} 2.675×10^{-2} 1.393×10^{-24}	1.866×10^{-3} 3.003×10^{-22} 4.936×10^{-152} 4.419×10^{-306}	2.021×10^{0} 6.776×10^{-4} 4.572×10^{-35} 2.914×10^{-246}	1.821×10^{-1} 9.747×10^{-12} 9.192×10^{-84} 6.120×10^{-588}	4.273×10^{-1} 2.920×10^{-9} 9.661×10^{-67} 1.120×10^{-469}	3.629×10^{-4} 2.696×10^{-29} 3.469×10^{-202} 1.231×10^{-1436}
Problem 2 $\beta_0 = 0.99$	2.728×10^{0} 2.681×10^{-4} 5.287×10^{-31} 6.124×10^{-218}	4.725×10^{2} 1.882×10^{1} 2.315×10^{-18} 2.537×10^{-134}	6.445×10^{-1} 1.055×10^{-10} 3.100×10^{-115} 2.257×10^{-207}	2.204×10^{-3} 1.033×10^{-4} 8.798×10^{-34} 2.845×10^{-237}	2.113×10^{-2} 2.889×10^{-11} 1.849×10^{-80} 3.148×10^{-565}	6.366×10^{-12} 7.106×10^{-8} 4.880×10^{-57} 3.220×10^{-410}	2.188×10^{-4} 7.824×10^{-20} 5.842×10^{-210} 5.755×10^{-1439}
Problem 3 $\beta_0 = 0.35$	8.081×10^{-5} 4.460×10^{-10} 6.959×10^{-207} 1.566×10^{-1444}	5.264×10^{-4} 1.102×10^{-23} 1.939×10^{-161} 1.209×10^{-239}	4.672×10^{-5} 3.713×10^{-101} 3.090×10^{-215} 2.218×10^{-388}	5.778×10^{-4} 9.745×10^{-12} 4.749×10^{-161} 3.372×10^{-312}	5.697×10^{-5} 4.725×10^{-125} 1.276×10^{-213} 1.341×10^{-610}	4.881×10^{-5} 1.322×10^{-111} 1.478×10^{-219} 4.745×10^{-1533}	3.831×10^{-6} 1.659×10^{-44} 1.854×10^{-270} 3.120×10^{-1889}
Problem 4 $\beta_0 = 1.5$	1.155×10^{0} 1.080×10^{-2} 1.161×10^{-18} 1.883×10^{-138}	3.607×10^{2} 3.462×10^{-13} 3.373×10^{-80} 6.776×10^{-57}	1.430×10^{-1} 1.891×10^{-12} 3.773×10^{-296} 4.795×10^{-692}	6.706×10^{-2} 2.723×10^{-13} 5.842×10^{-90} 1.222×10^{-650}	3.620×10^{-1} 3.787×10^{-13} 1.211×10^{-502} 4.141×10^{-657}	9.101×10^{-2} 1.234×10^{-15} 3.262×10^{-593} 2.216×10^{-781}	9.686×10^{-3} 1.376×10^{-13} 1.933×10^{-669} 9.351×10^{-902}

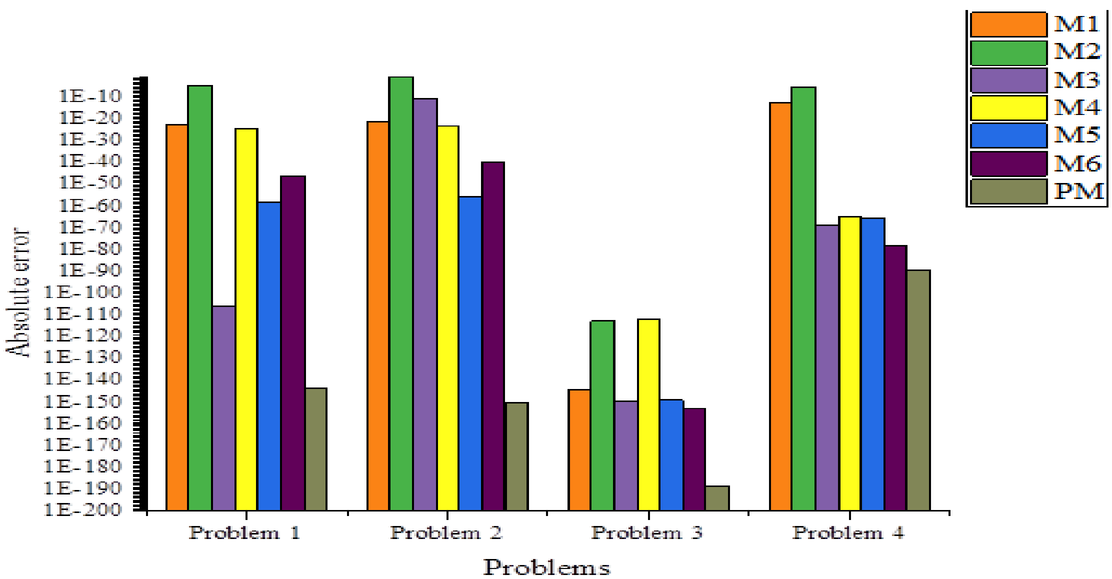

Fig. 1: Graphical representation of the 4th iteration of problem 1 to 4 of Table 2

[10] A. Zein, "A general family of fifth-order iterative methods for solving nonlinear equations," European Journal of Pure and Applied Mathematics, vol. 16, no. 4, pp. 2323–2347, 2023, doi: 10.29020/nybg.ejpam.v16i4.4949.

[11] U. K. Qureshi, S. Jamali, Z. A. Kalhoro, and A. G. Shaikh, "Modified quadrature iterated methods of Boole Rule and Weddle Rule for solving non-linear equations," Journal of Mechanics of Continua and Mathematical Sciences, vol. 16, no. 2, pp. 87–101, 2021, doi: 10.26782/jmcms.2021.02.00008.

[12] B. Neta, "A new derivative-free method to solve nonlinear equations," Mathematics, vol. 9, no. 6, pp. 1–5, 2021, doi: 10.3390/math9060583.

[13] J. Li, X. Wang, and K. Madhu, "Higher-order derivative-free iterative methods for solving nonlinear equations and their basins of attraction," Mathematics, vol. 7, no. 1, pp. 1–15, 2019, doi: 10.3390/math7111052.

[14] H. M. S. Bawazir, "Fourth, fifth and seventh-order iterative methods for solving nonlinear equations," Hadhramout University Journal of Natural and Applied Sciences, vol. 21, no. 1, pp. 1–10, 2024.

[15] S. Thota and P. Shanmugasundaram, "On new sixth and seventh order iterative methods for solving nonlinear equations using homotopy perturbation technique," BMC Research Notes, vol. 15, p. 267, 2022, doi: 10.1186/s13104-022-06154-5.

[16] S. Shams, M. Mudassir, N. Kausar, and I. A. Șomîtcă, "Efficient multiplicative calculus-based iterative scheme for nonlinear engineering applications," Mathematics, vol. 12, no. 22, p. 3517, 2024.

[17] H. I. Siyyam, "A note on some higher-order iterative methods free from second derivative for solving nonlinear equations," International Mathematical Forum, vol. 6, pp. 3381–3386, 2011.

[18] M. Fardi, M. Ghasemi, E. Kazemi, and R. Ezzati, "Seventh-order iterative algorithm free from second derivative for solving algebraic nonlinear equations," International Journal of Industrial Mathematics, vol. 5, no. 1, pp. 1–5, 2013.

[19] M. A. Hafiz and S. M. H. Al-goria, "New ninth- and seventh-order methods for solving nonlinear equations,"

European Scientific Journal, vol. 8, no. 27, pp. 83–95, 1857.

[20] O. S. Solaiman and I. Hashim, "Two new efficient sixth-order iterative methods for solving nonlinear equations," Journal of King Saud University – Science, vol. 31, no. 4, pp. 701–705, Oct. 2019, doi: 10.1016/j.jksus.2018.03.021.

[21] S. Jamali, Z. A. Kalhoro, and I. Q. Memon, "An efficient four-step fifteenth order method for solution of non-linear models in real-world problems," Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences, vol. 61, no. 3, pp. 273–281, Sep. 2024, doi: 10.53560/PPASA(61-3)852.

[22] Z. Abbasi, Z. A. Kalhoro, S. Jamali, A. W. Shaikh, and O. A. Rajput, "A novel approach for real-world problems based on Hermite interpolation technique and analysis using basins of attraction," Science, vol. 5, no. 3, pp. 112–126, 2024.

[23] S. Jamali, Z. A. Kalhoro, A. W. Shaikh, M. S. Chandio, and S. Dehraj, "A new three-step derivative-free method using weight function for numerical solution of non-linear equations arises in application problems," VFAST Transactions on Mathematics, vol. 10, no. 2, pp. 164–174, 2022, doi: 10.21015/vtm.v10i2.1289.

[24] A. Naseem, M. A. Rehman, and J. Younis, "Some real-life applications of a newly designed algorithm for nonlinear equations and its dynamics via computer tools," Complexity, vol. 2021, 2021, doi: 10.1155/2021/9234932.

[25] A. Naseem, M. A. Rehman, and J. Younis, "A new root-finding algorithm for solving real-world problems and its complex dynamics via computer technology," Complexity, vol. 2021, 2021, doi: 10.1155/2021/6369466.