NUMERICAL SOLUTION OF SINGULAR TWO-POINT BOUNDARY VALUE PROBLEMS USING GALERKIN'S FINITE ELEMENT METHOD

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ABSTRACT

Galerkin's finite element method is presented for numerical solution of singular two-point boundary value problems (BVPs) for certain ordinary differential equation having singular coefficients. These problems arise while reducing partial differential equations to ordinary differential equations by physical symmetry. The numerical results show the capability of Galerkin's finite element method, to remove the difficulty of convergence due to singularity for singular BVPs. Some examples are given to demonstrate the effectiveness of the method and comparison of the numerical results made with the exact solutions.

Keywords: Galerkin Method, Finite Element Method, Ordinary Differential Equations, Singular Points, Boundary Value Problems.

1. INTRODUCTION

Galerkin's Finite Element Method (FEM) is one of the computational techniques for obtaining approximate solutions to many complicated problems that would be intractable by other techniques [1, 2]. The method involves dividing the domain of solution into a finite number of simple sub-domains, the finite elements. It is a well established numerical technique. In this paper, the method of weighted residual in Galerkin's finite element formulation is used for obtaining smooth approximations to the solution of a system of second-order singular two-point boundary value problems (BVPs) of the type:

$$y''(x)+\frac{1}{x}y'(x)+q(x)y(x)=f(x), \qquad (1)$$

subject to the boundary conditions

$$y(a)=\alpha_1 \quad \text{and} \quad y(b)=\alpha_2 \qquad (2)$$

where $q(x)$ and $f(x)$ are continuous functions on $(0,1]$ and α_1, α_2 are real constants. These problems (1)–(2) are generally encountered in many areas of science and engineering e.g., in the fields of fluid mechanics, elasticity, reaction–diffusion processes, chemical kinetics and other branches of applied mathematics [3]. Numerical solutions of these problems are of great importance due to its wide application in scientific research. Singular BVPs have been studied by several researchers. Convergence difficulties have been faced due to the singularity at $x=0$ on the left side of the differential equation (1). Attempts by many researchers for the removal of singularity are based on using the series expansion procedures in the neighborhood $(0,\delta)$ of singularity (δ is vicinity of the singularity) and then solve the regular boundary value problem in the interval $(\delta,1)$ using any numerical method. Kamel Al-Khaled [4] used the Sinc-Galerkin method and homotopy-perturbation method (HPM) to search for approximate solutions of a certain class of singular two-point BVPs. Junfeng Lu [5] used variational iteration method (VIM) to solve two-point BVPs.

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Abu-Zaid et al [6] provided a finite difference approximation to the solution of the above problems. A method based on cubic splines for solving a class of singular two-point BVPs was presented by Ravi Kanth and Reddy [7]. Recently, Bataineh [8] extended application of the modified homotopy analysis method for singular two-point boundary value problems and Ravi Kanth [9] demonstrates the differential transform method for the solution the same problems. The existence of a unique solution of (1)–(2) was discussed in [6, 10].

The aim of this paper is to introduce a numerical technique as an alternative to the existing numerical methods to remove the difficulty of convergence due to singularity at $x=0$ in solving singular two-point BVPs and solutions are computed in the entire domain. Once the solution has been computed, the information required for FEM interpolation between mesh points is available for the entire solution domain. This is particularly important when the solution of the boundary-value problem is required at different locations in the solution interval $[a,b]$.

The paper is organized as follows. In Section 2, the Galerkin's finite element formulation is formulated using linear Lagrange polynomial. Comparison with exact solutions and discussion regarding results of three examples is given in Section 3 and 4.

2. GALERKIN'S FINITE ELEMENT FORMULATION

The Galerkin's method [1, 2, and 12] in the finite element context requires that we choose a suitable trial or basis function that is applied locally over a typical finite element in the complete x domain. Let us denote this trial function by $\bar{y}$. In this case it is necessary to satisfy inter-element compatibility with respect to displacements. In other words the trial function is C^0-continuous. Each element has two nodes. We interpolate the function at each node of the element. This requires one unknown parameters at each node of the element. Let the unknown trial function $\bar{y}=a_1+a_2x$ for any arbitrary element of the discretized region. Rather than formulating the problem in terms of arbitrary constants a_1 and a_2, we prefer to express the linear trial function in terms of values of the dependent functions at nodes i and j (the convention used by Zienkiewicz, Stasa and Shaukat Iqbal et al. [1, 2 and 12]).

$$\bar{y}(x)=N_1y_i+N_2y_j \tag{3}$$

$$\bar{y}(x)=[\mathbf{N}]\{\mathbf{y}\} \tag{4}$$

Here $\{\mathbf{y}\}^T=\begin{bmatrix}y_i & y_j\end{bmatrix}$ and $[\mathbf{N}]=\begin{bmatrix}N_1 & N_2\end{bmatrix}$ are the matrices of interpolation functions and $N_1=x_j-x/x_j-x_i$ & $N_2=x-x_i/x_j-x_i$, the trial function constants now are the nodal variables of the dependent variable $\bar{y}$.

Now, the governing differential equation (1) can be written as:

$$\frac{d}{dx}(x\frac{dy}{dx})+xq(x)y=xf(x) \tag{5}$$

Galerkin's finite element formulation as given in [1, 2, and 12], is used for our particular problem and after substituting the trial functions, the equation (5) can be written in discretized form as:

$$wx\bar{y}'\Big|_{X_1}^{X_2}-\sum_{e=1}^{n}\left[\int_{X_e}xw'\bar{y}'dx-\int_{X_e}xwq(x)\bar{y}dx+\int_{X_e}xwf(x)dx\right]=0 \tag{6}$$

where $'e'$ represents the element and $'n'$ represents the total number of elements in the discretized region. Writing equation (6), in matrix form, we obtain:

$$\sum_{e=1}^{n}\left(\int_{X_e}x\left[\mathbf{N}'\right]^T\left[\mathbf{N}'\right]\{y\}dx-\int_{X_e}xq(x)[\mathbf{N}]^T[\mathbf{N}]\{y\}dx\right) = [\mathbf{N}]^T x\bar{y}'\Big|_{X_1}^{X_2}-\sum_{e=1}^{n}\int_{X_e}x[\mathbf{N}]^T f(x)dx \tag{7}$$

The equations for the elements must combine in such a manner that only the boundary terms for element nodes on the region boundary will contribute; all other terms for interior nodes will be zero. This implies that the boundary terms for elements at common interior nodes cancel each other. Therefore, equation (7) in matrix notation can be written as:

$$[\mathbf{K}]\{\mathbf{y}\}=\{\mathbf{F}\}+\{\mathbf{Q}\} \tag{8}$$

where $\mathbf{K}$ is a stiffness matrix, $\mathbf{F}$ is a force vector and $\mathbf{Q}$ is a vector regarding boundary conditions.

3. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we illustrate the numerical scheme by three singular two-point boundary value problems, which have been discussed in literature [4, 5, 8, 9 and 11].

Example 1.

First we consider the following singular two-point boundary value problem [8, 9 and 11]:

$$y''(x)+\frac{1}{x}y'(x)+y(x)=f(x) \tag{9}$$

where

$$f(x)=4-9x+x^2-x^3 \quad 0<x\leq 1, \tag{10}$$

subject to the boundary conditions

$$y(0)=0 \quad \text{and} \quad y(1)=0. \tag{11}$$

Equation (15) can be formed as:

$$\frac{d}{dx}(x\frac{dy}{dx})+xy-xf(x)=0. \tag{12}$$

The exact solution of (9) subject to (11) in this case is $y(x)=x^2-x^3$. Now, applying the procedure given in section 2, we will get:

$$wx\bar{y}'\Big|_{x_1}^{x_2}-\int_X xw'\bar{y}'dx+\int_X xw\bar{y}dx-\int_X wx(4-9x+x^2-x^3)dx=0 \tag{13}$$

Example 2.

Consider the following singular two-point boundary value problem [5, 8 and 9]:

$$y''(x)+\frac{1}{x}y'(x)+y(x)=f(x) \tag{14}$$

where

$$f(x)=\frac{5}{4}+\frac{x^2}{16} \quad 0<x\leq 1, \tag{15}$$

subject to the boundary conditions

$$y(0)=1 \quad \text{and} \quad y(1)=17/16. \tag{16}$$

Equation (14) can be formed as:

$$\frac{d}{dx}(x\frac{dy}{dx})+xy-xf(x)=0. \tag{17}$$

The exact solution of (14) subject to (16) in this case is $y(x)=1+\frac{x^2}{16}$. Now, applying the procedure given in section 2, we will get:

$$wx\bar{y}'\Big|_{x_1}^{x_2}-\int_X xw'\bar{y}'dx+\int_X xw\bar{y}dx-\int_X wx(\frac{5}{4}+\frac{x^2}{16})dx=0 \tag{18}$$

Example 3.

Finally we consider the following singular two-point boundary value problem [4, 8 and 9]:

$$(1-\frac{x}{2})y''(x)+\frac{3}{2}(\frac{1}{x}-1)y'(x)+(\frac{x}{2}-1)y(x)=f(x) \tag{19}$$

where

$$f(x)=5-\frac{29x}{2}+\frac{13x^2}{2}+\frac{3x^3}{2}-\frac{x^4}{2} \quad 0<x\leq 1, \tag{20}$$

subject to the boundary conditions

$$y(0)=0 \quad \text{and} \quad y(1)=0. \tag{21}$$

Equation (19) can be formed as:

$$\frac{d}{dx}\left[(2x-x^2)\frac{dy}{dx}\right]+(1-x)\frac{dy}{dx}+(x^2-2x)y-2xf(x)=0. \tag{22}$$

The exact solution of (19) subject to (21) in this case is $y(x)=x^2-x^3$. Now, applying the procedure given in section 2, we get:

$$w(2x-x^2)y'\Big|_{x_1}^{x_2}-\int_X(2x-x^2)w'y'dx+\int_X(1-x)wy'dx+\int_X(x^2-2x)wydx$$
$$-\int_X wx(10-29x+13x^2+3x^3-x^4)dx=0 \tag{23}$$

The numerical results for examples 1, 2 & 3 are shown in Tables 1, 2 & 3 for 40 elements ($h=1/40$). The computed solutions compare very well with the exact solutions at various values of x. The numerical results presented in tables 1, 2 & 3 clearly show the existence of the solution at singular points, which reflects the potential of the Galerkin's finite element method. Due to the singularity at $x=0$ in the given examples, people neglect the effect of singularity on the solution and makes calculations in the immediate neighborhood of the singular point. Convergence difficulties faced by other numerical methods have been removed by formulating the singular BVPs by Galerkin's finite element method. These examples have been considered by Cui Geng [11], Junfeng [5] and Al-Khaled [4]. Figure. 1 shows the convergence of the maximum error for three examples with the decrease in the step size (decrease in element size) or increase in the number of elements. Results indicate the formulation is accurately approximating the solution.

4. CONCLUSIONS

Solutions of singular two-point boundary value problems have been investigated for certain ordinary differential equation having singular coefficients using Galerkin's finite element formulation. This method enables us to approximate the solution at every point of the domain of the problem. Convergence difficulties faced by other numerical methods due to singularity have been removed by formulating the singular BVPs by Galerkin's finite element method. The results obtained are very encouraging and FEM performs better than other existing numerical methods. Examples demonstrate that the numerical results of the finite element method are generally very accurate and in excellent agreement with the exact solution. The numerical results which are presented in Tables reinforce the conclusions made by many researches that the efficiency of the finite element method gives it much wider applicability.

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Table 1: Numerical Results for 40 elements of Example 1

$(h = 1/40)$

x	FEM	Exact	Error
0.0	0	0	0
0.1	9.1672E-3	9.0000E-3	1.6719E-4
0.2	3.2175E-2	3.2000E-2	1.7522E-4
0.3	6.3165E-2	6.3000E-2	1.6526E-4
0.4	9.6148E-2	9.6000E-2	1.4762E-4
0.5	1.2513E-1	1.2500E-1	1.2581E-4
0.6	1.4410E-1	1.4400E-1	1.0162E-4
0.7	1.4708E-1	1.4700E-1	7.6175E-5
0.8	1.2805E-1	1.2800E-1	5.0320E-5
0.9	8.1025E-2	8.1000E-2	2.4734E-5
1.0	0	0	0

Table 2: Numerical Results for 40 elements of Example 2

$(h = 1/40)$

x	FEM	Exact	Error
0.0	1.0	1.0	0
0.1	1.000626	1.000625	1.1074E-6
0.2	1.002501	1.002500	1.2826E-6
0.3	1.005626	1.005625	1.3221E-6
0.4	1.01000128	1.01	1.2827E-6
0.5	1.01562618	1.015625	1.1826E-6
0.6	1.022501	1.022500	1.0308E-6
0.7	1.0306258	1.0306250	8.3276E-7
0.8	1.04000059	1.0400000	5.9263E-7
0.9	1.05062531	1.05062500	3.1395E-7
1.0	1.0625	1.0625	0

Table 3: Numerical Results for 40 elements of Example 3

$(h = 1/40)$

x	FEM	Exact	Error
0.0	0	0	0
0.1	9.0723E-3	9.0000E-3	7.2331E-5
0.2	3.2063E-2	3.2000E-2	6.3038E-5
0.3	6.3052E-2	6.3000E-2	5.2129E-5
0.4	9.6042E-2	9.6000E-2	4.1548E-5
0.5	1.2503E-1	1.2500E-1	3.1753E-5
0.6	1.4402E-1	1.4400E-1	2.2924E-5
0.7	1.4702E-1	1.4700E-1	1.5174E-5
0.8	1.2801E-1	1.2800E-1	8.6319E-6
0.9	8.1003E-2	8.1000E-2	3.4807E-6
1.0	0	0	0

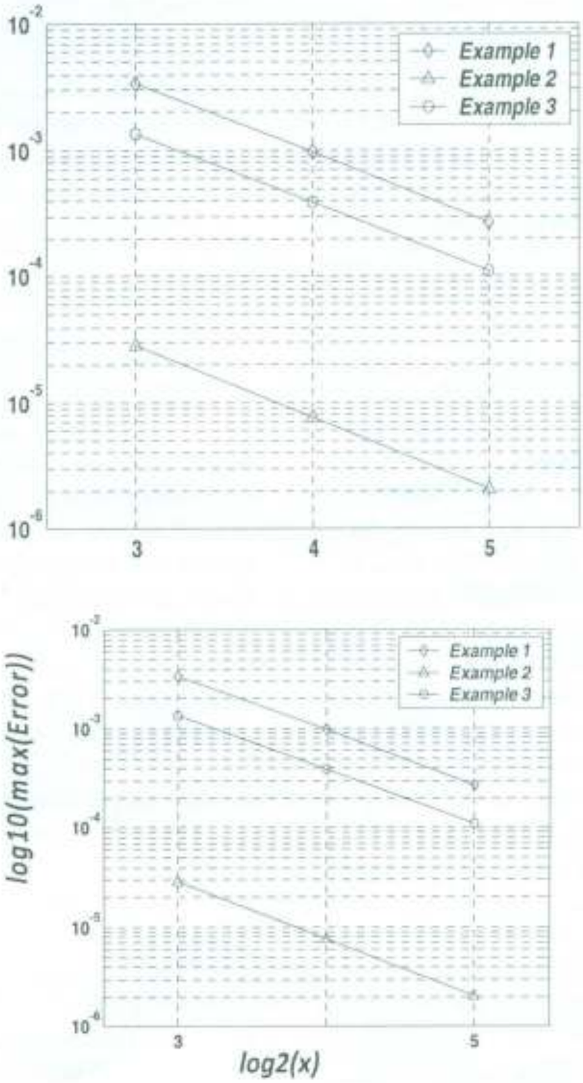

Figure 1: Error convergence with decreasing step size or increasing