AN OPTIMALLY ACCURATE SPECTRAL VOLUME FORMULATION WITH SYMMETRY PRESERVATION

Fareed Hussain Mangi*, Umair Ali Khan**, Intesab Hussain Sadhayo**, Rameez Akbar Talani***, Asif Ali Memon*

ABSTRACT

High order accuracy has become a challenge in numerical simulations for engineering applications. A number of methods have been developed which provide adequate levels of accuracy in numerical simulations. Among them, the Spectral Volume (SV) method is the most effective approach that ensures local conservation and achieves high order accuracy for unstructured grids. A number of formulations for SV method have been reported and tested. However, these formulations are inconsistent and/or do not preserve the symmetry property of an elliptical operator. In this paper, we propose a new formulation, named as Symmetry Preservation in Spectral Volume (SPSV) for spectral volume method that is not only consistent and stable, but also preserves the symmetry property of an elliptical operator. We test the newly proposed formulation on diffusion equation and Burger equation using different boundary conditions and evaluate it with respect to symmetry preserving capability, accuracy, and stability. Furthermore, we also present a detailed comparison between the local SV and the SPSV formulation. Our results show that besides preserving the symmetry of an elliptical operator, the SPSV formulation is more accurate and stable.

Keywords: *spectral volume method; symmetry preserving; diffusion equation.*

1. INTRODUCTION

The Spectral Volume (SV) [1-3] and Discontinuous Galerkin (DG) [4] are two popular high-order methods for hyperbolic conservation laws capable of handling unstructured grids. The critical analysis of DG and SV methods show that they both can achieve high order accuracy on unstructured grids, but take more memory as compared to the conventional second-order method. They are also similar in efficiency. The SV method appears to have a higher resolution for discontinuities because the data limiting can be done at the sub-element level. The SV method is based on two major points: (i) flux reconstruction at the interface taking into account the physics of the problem under consideration, and (ii) reconstructing high-order polynomials within spectral volumes to achieve higher accuracy. SV method particularly differs from other methods, e.g., k-exact finite volume and its variants [7-12] with respect to data reconstruction. SV method does not consider a large number of adjacent cells to reconstruct a polynomial. Instead, it divides a simple grid cell into a number of sub-cells. The average values of these sub-cells which represent the solution unknown are the inputs to the polynomial reconstruction process. An appropriate approximation approach, e.g., Riemann solver can be used to compute the numerical flux with the formulated solutions at both sides of a given interface. In the subsequent step, the traditional finite volume method [13] can be used to update the solution unknowns to higher accuracy.

The spectral volume method comprising the above mentioned steps is known to be effective for hyperbolic conservation laws, including nonlinear systems on unstructured grids [1-3]. It can successfully solve nonlinear time-dependent hyperbolic systems of conservation laws using a number of schemes including explicit method, Runge Kutta time integration [14], Riemann method, and total variation bounded non-limiters [15]. The ability of SV method to deliver high accuracy, geometrical flexibility and conservation has been confirmed by a number of studies. The spectral volume method has also been successfully applied for the Euler's equations. In order to extend it to the Navier-Stokes equations, three different formulations have been tested by Yuzhiet al. [16] for the diffusion term: (i) naïve implementation on diffusion equation, (ii) implementation of Galerkin method [17], and (iii) inclusion of a penalty factor in the numerical flux which is similar to the methods described in [18-20]. The first formulation is found to be inconsistent in a number of numerical analysis. On the other hand, although the local SV and the penalty SV formulations are consistent and stable, there are still some drawbacks that need to be improved. For the local SV formulation, the symmetry property of an elliptic operator is not preserved. Consequently the numerical solution to a symmetric physical problem will not be symmetric. For the penalty SV formulation, the order of accuracy of the method may be the same as the order of the polynomial used for the reconstruction. For example, with quadratic reconstruction, only second order

* Department of Energy & Environment Engineering,

** Department of Computer Systems Engineering,

*** Department of Electrical Engineering

Quaid-e-Awam University of Engineering, Science & Technology, Nawabshah.
email: fareedmangi@gmail.com, umair.khan@quest.edu.pk

accuracy can be achieved for diffusion equation, while third order accuracy can be obtained for first order equation.

In this paper, a new formulation for the SV method is proposed which preserves the symmetry property of the elliptic operator. In addition, with degree of polynomial reconstruction, we can obtain order accuracy for convection-diffusion equation on uniform grid. This formulation is referred to as Symmetry Preservation in Spectral Volume (SPSV) scheme for SV method.

This paper is structured as follows. Section 2 gives a brief account of the spectral volume method. Section 3 describes the traditional local SV formulation. In section 4, we discuss our proposed SV method, referred to as Symmetry Preservation in Spectral Volume (SPSV) scheme. Section 5 provides an in-depth accuracy analysis, discussion on stability issues, and numerical evaluation of the proposed scheme for diffusion equation and Burger equation along with comparisons with the local SV formulation. Section 6 concludes the paper.

2. SPECTRAL VOLUME METHOD

We first briefly discuss the principle of the spectral volume proposed by Wang [1]. For simplicity, only one-dimensional case will be considered in this paper. Let us consider a scalar convection-diffusion equation.

$$\frac{du}{dt} + \frac{df(u(x,t))}{dx} = \frac{d}{dx}(\mu \frac{du}{dx}), \qquad in[a,b]x[0,T] \tag{1}$$

Given a partition of the domain $[a,b]$, $\{x_{i+1/2}\}_{i=0}^{N}$, the domain is then divided into N non-overlapping spectral volumes, i.e.,

$$[a,b] = \cup_{i=1}^{N} S_i, \quad S_i = [x_{i-1/2}, x_{i+1/2}] \tag{2}$$

Let $h_i = x_{i+1/2} - x_{i-1/2}$, and denote the quantity $\max_{1\le i\le N} h_i$ by h. Given a desired numerical order of accuracy k for equation (1), each spectral volume S_i is subdivided into k Control Volumes (CV) using the following partitioning with $\{x_{i+1/2}\}_{j=0}^{k}$ with $x_{i,1/2} = x_{i-1/2}$ and $x_{i,k+1/2} = x_{i+1/2}$. The j^{th} CV of S_i is then given by equation (3).

$$C_{i,j} = (x_{i,j-1/2}, x_{i,j-1/2}), \quad j = 1,\dots,k \tag{3}$$

The state variable representing the average values of cells at time t for $C_{i,j}$ is given as:

$$\bar{u}_{i,j}(t) = \frac{\int_{x_{i,j-1/2}}^{x_{i,j+1/2}} u(x,t)dx}{h_{i,j}}, \quad j = 1,\dots,k, \; i = 1,\dots,N \tag{4}$$

We employ the Lagrange interpolation polynomial to reconstruct the solution inside each SV as follows.

$$p_i(x,t) = \sum_{j=1}^{k} u_{i,j}(t) h_j(x) \tag{5}$$

with

$$h_j(x) = \prod_{s=1, s\neq j}^{k} \frac{x - x_s^c}{x_j - x_s^c} \tag{6}$$

Where $x_s^c = \frac{1}{2}(x_{i,s-1/2} + x_{i,s+1/2})$ is the center of the control volume and $u_{i,j}(t)$ is the value of the reconstruction function $p_i(x,t)$ at the center of the CV. Integration of the $p_i(x,t)$ function over each CV gives a linear equation between $u_{i,j}(t)$ and $\bar{u}_{i,j}(t)$:

$$\begin{aligned} &\int_{x_{I,J-/2}}^{x_{I,J+/2}} p_i(x,t)\,dx = \Sigma \int_{x_{I,J-/2}}^{x_{I,J+/2}} u_{i,j}(t) h_j(x)\,dx = \\ &\Sigma u_{i,j}.\alpha_j = \bar{u}_{i,j}(t) \qquad j = 1,2,..k \end{aligned} \tag{7}$$

Inversion of the above function gives the reconstruction function expressed in terms of $\bar{u}_{i,j}$. It must be noted that just like finite element method, inversion of the above system needs not to be done for each SV. It can be computed once using a mapping between a standard SV and a general SV.

This reconstruction can be solved in many ways. A unique polynomial of degree of at most $k-1$, whose cell average in each of the CVs in S_i agrees with that of u(x), i.e.,

$$\bar{u}_{i,j} = \frac{\int_{x_{i,j-1/2}}^{x_{i,j+1/2}} p_i(x)dx}{h_{i,j}}, \quad j = 1,\dots,k \tag{8}$$

This polynomial $p_i(x)$ is the k^{th} order approximation we are looking for as long as the function u(x) is smooth in the region covered by S_i. Ultimately, we need to compute the approximate solutions of u(x) at the CV boundaries to update the state variables at the next time level. Since the mappings from the given cell averages $\bar{u}_{i,j}$ to the CV boundary values are linear, there exist coefficients c_{jl} which depend on the order of accuracy k, and on the mesh size $h_{i,j}$ in S_i, but not on the function u itself, such that,

$$u_{i,j+\frac{1}{2}} = \sum_{l=1}^{k} c_{jl} \bar{u}_{il}, j = 0,\dots,k \tag{9}$$

The detailed discussion for $c_{jl}$is given in [1]. For equal-distance partition, it is given by:

$$c_{jl} = \sum_{r=l}^{k} \frac{1}{\prod_{q=0, q\neq r}^{k} (r-q)} \sum_{m=0, m\neq r}^{k} \prod_{q=0, q\neq r,m}^{k} (j-q) \tag{10}$$

With the reconstructed state variables at the CV boundaries for all SVs, we can update each control volume as if it is independent, i.e.,

$$\frac{du_{i,j}}{dt}h_{i,j} + (f_{i,j+1/2} - f_{i,j-1/2}) = \mu[(u_x)_{i,j+1/2} - (u_x)_{i,j-1/2})], \quad (11)$$

where $f_{i,j+1/2}$ is the numerical fluxe at $x_{i,j+1/2}$. We perform time integration using 3rd order Runge-Kutta method [4] for the diffusion equation and implicit method with deffered correction is used for the Burger's eqation.

3. LOCAL SPECTRAL VOLUME FORMULATION

In the local SV formulation, equation (1) is first transformed to a first order system by introducing a new variable $q = u_x$. The resulting system is given by,

$$\begin{aligned} \frac{du}{dt} + \frac{df(u(x,t))}{dx} &= \mu\frac{dq}{dx} \\ q &= \frac{du}{dx} \end{aligned} \quad (12)$$

Integrating equation (12) over each control volume, we obtain:

$$\begin{aligned} h_{i,j}\frac{du_{i,j}}{dt} + [\hat{f}_{i,j+1/2} - \hat{f}_{i,j-1/2}] &= \mu[\hat{q}_{i,j+1/2} - \hat{q}_{i,j-1/2}] \\ q_{i,j}.h_{i,j} &= \hat{u}_{i,j+1/2} - \hat{u}_{i,j-1/2} \end{aligned} \quad (13)$$

At the boundary of the SV, the solution is discontinuous. We have to determine the numerical flux $f_{i,1/2}$ in terms of its left state and right state, namely,

$$f_{i-1,k+1/2} = f_{i,1/2} = f(u_{i-1,k+1/2}, u_{i,1/2}) \quad (14)$$

Numerical method developed in finite volume method, such as approximated Riemann solver, can be applied for this purpose. For the scalar equation investigated in the present study, upwind scheme is applied to compute the numerical flux $f_{i,j+1/2}$. The computation of the numerical flux $\hat{q}$ and $\hat{u}$ is specific to the local SV formulation. They are chosen according to [16].

$$\begin{aligned} \hat{u}_{i,1/2} &= u_{i,1/2} \\ \hat{q}_{i,1/2} &= q_{i-1,k+1/2} \end{aligned} \quad (15)$$

The selection of upwind/downwind for u and q can be choosen alternatively. Third-order TVD Runge-Kutta method [5] is applied for time integration. Let k be the degree of the reconstruction polynomial. Numerical solutions are computed at $t = 0.7$ for the five cases:$k = 1$ (linear reconstruction),$k = 2$(quadratic reconstruction), $k = 3$(cubic reconstruction), $k = 4$ (quartic reconstruction), and $k = 5$ (quintic reconstruction). The L1 and L_∞ errors and observed orders of accuracy are presented in the comparison tables and graghs, from which we note that a $(k+1)^{th}$ order of accuracy is achieved for a degree m polynomial reconstruction. No discontinuity is present at the internal interfaces of the CV inside a SV. At these internal interfaces, numerical flux are computed directly using the reconstruction function.

4. SYMMETRY PRESERVATION IN SPECTRAL VOLUME (SPSV) METHOD

In the SPSV formulation, equation (1) is directly interaged over each CV, which gives:

$$u_{i,j}(t).h_{i,j} + [f_{i,j+1/2} - f_{i,j-1/2}] = \mu\left[\frac{du}{dx}|_{i,j+1/2} - \frac{du}{dx}|_{i,j-1/2}\right] \quad (16)$$

Solution unknowns and the fluxes are computed using two sets of points, i.e., solution points and flux points. Figure 1 depics these two sets of points. The conserved variables at the solution points represent the solution unknown. Whereas, the fluxes are computed at the flux points in order to update the solution unknowns. With k solution unknowns, a polynomial of $(k-1)$ degree can be reconstructed.

For our case, the flux points are given as:

$$x_{i+1/2} = \frac{i*[b-a]}{k*N}, \quad i = 0,1,,,k \quad (17)$$

Whereas solution points are given as:

$$x_i^c = \frac{1}{2}(x_{i+1/2} + x_{i-1/2}), \quad i = 1,2,,,k \quad (18)$$

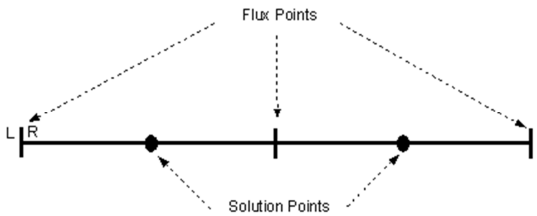

Fig. 1: Linear Spectral Volume

Based on the solution unknowns at the solution points, the solution in the spectral volume is reconstructed with a $(k-1)$ degree Lagrange interpolation polynomial given by,

$$h_j(x) = \prod_{s=1_{s\neq j}}^{k} \{\frac{x-x_s}{x_j-x_s}\} \quad (19)$$

This reconstruction function is referred to as the primary reconstruction function. To compute the viscous flux, we construct a new k degree Lagrange polynomial based on the conserved variables at the flux points.

$$l_{j+1/2}(x) = \prod_{\substack{s=0 \\ s\neq j}}^{N} \left\{\frac{x - x_{s+1/2}}{x_{j+1/2} - x_{s+1/2}}\right\} \quad (20)$$

This new reconstruction function is reffered to as auxilary reconstruction function. The idea of using an auxilary reconstruction function was first proposed in [21] for a staggered-grid multi-domain spectral method and adapted by Wang et al. in a spectral difference method [22].

The following steps are used to compute the inviscid flux:

1. Given the cell-averaged value $\bar{u}_i$ for conserved variable for each control volume, conserved variables at the flux points are computed using equation (9).
2. The inviscid fluxes are computed directly using the flux function because the solution is continuous.
3. The inviscid fluxesare computed at the spectral volume interfaces based on the left and the right states using an appropriate flux reconstruction scheme such as a Riemann solver. As we consider a scalar equation here, an upwinding scheme without limiter is applied.

The computation of viscous flux comprises the following steps.

1. The conserved variables at the flux points are computed using the primary reconstruction function.
2. The solutions at both sides of the SV interface are used to compute the average solution, i.e.,

$$\bar{u} = \frac{1}{2}(u_L + u_R)$$

The solution is updated at the boundary of the computational domain with boundary condition. Then, the auxilary reconstruction function is computed.

3. The gradient at the cell interface flux points is computed using the average of the gradients at both sides of the SV interface, i.e.,

$$\nabla\bar{u} = \frac{1}{2}(\nabla\bar{u}_L + \nabla\bar{u}_R)$$

Here, the gradients on both sides of the interface are computed with the auxilary reconstruction function.

At internal interface (control volume interface inside the SV) two choices are possible.

1. $u_x|_{i,j+1/2}, \quad j = 1,2,\ldots,k-1$ computed with the auxilary reconstruction function. This is reffered to as A-Scheme.
2. $u_x|_{i,j+1/2}, \quad j = 1,2,\ldots,k-1$ computed with the primary reconstruction function. This is referred to as P-Scheme.

In persuance of higher accuracy, we apply A-scheme for odd order and P-scheme for even order.

5. NUMERICAL EVALUATION

We apply the proposed technique on diffusion equation and Burger's equation.

5.1. Diffusion Equation

Let us consider one-dimensional diffusion equation,

$$u_t = u_{xx}, \qquad x \in [0,2\pi] \quad (21)$$

with initial conditions $u(x,0) = \sin(x)$. The numerical analysis for every formulation is performed with periodic and Dirichlet boundary conditions. The exact solution of diffusion equation is given as under.

$$u(x,t) = e^{-t}\sin(x)$$

We perform a number of tests in order to evaluate these two mentioned formulations. The results of these tests are described explicitly under different headings.

5.1.1. Symmetry Problem

Diffusion equation contains the elliptical operator and other numerical methods lose symmetry when solving such type of equations. A numerical method must have the symmetry conserving property to achieve good accuracy. To check out the symmetry of the schemes, we select a third order SV method as a reference case. The domain is divided into 40 grid points. The comparison of these formulations is made with the exact solution. For simplicity, the results of first three and last two points are given in Table 1 and Table 2. It is evident that for both boundary conditions, only SPSV preserves the anti-symmetry features of the exact solution.

Table 1: Comparison with periodic boundary conditions

x	Exact Solution	SPSV Formulation	Local SV Formulation
0.026179	0.0130004	0.0129884	0.0129826
0.078539	0.0389656	0.0389656	0.0389775
0.130899	0.0648240	0.0648358	0.0648300
6.204645	-0.0389656	-0.0389656	-0.0389538
6.257005	-0.0130004	-0.0129884	-0.0129945

Table 2: Comparison with Dirichlet boundary conditions

x	Exact Solution	SPSV Formulation	Local SV Formulation
0.026179	0.0130004	0.0129884	0.0129737
0.078539	0.0389656	0.0389656	0.0389687
0.130899	0.0648240	0.0648358	0.0648213
6.204645	-0.0389656	-0.0389656	-0.0389746
6.257005	-0.0130004	-0.0129884	-0.0129435

The results given in Table 1 and Table 2 clearly show that the SPSV formulation preserves the anti-symmetry feature of the exact solution with both types of boundary condition.

5.1.2. Accuracy Analysis

Space discretization uptil sixth order is done for the presented formulations. OASP formulation with the optimalaccuracy condition is used here. Courant-Friedrich-Levy (CFL) number is chosen as high as possible for each order of discretization. Instead of all the results only L1 error for periodic boundary condition and L1 error for Dirichlet boundary condition with their respective achieved order are presented. To check the accuracy, SPSV formulation with optimal accuracy condition is compared with the local SV formulation. The accuracy up to 3rd order with different CFL and boundary conditions is presented in Table 3-6.

Table 3: 2nd order linear SV with CFL=0.6 (periodic boundary condition)

	Local SV Formulation				SPSV Formulation			
H	L1 Eerror	L1 Order	L_∞error	L_∞order	L1 Eerror	L1 Order	L_∞error	L_∞order
10	8.36E-3		2.29E-2		6.22E-3		9.82E-3	
20	1.99E-3	2.065	5.91E-3	1.9561	1.73E-3	1.843	2.72E-3	1.852
40	4.90E-4	2.0248	1.47E-3	1.998	4.49E-4	1.949	7.05E-4	1.947
80	1.22E-4	2.007	3.70E-4	1.999	1.13E-4	1.986	1.78E-4	1.986
160	3.04E-5	2.001	9.25E-5	1.999	2.84E-5	1.996	4.46E-5	1.996
320	7.61E-6	2.000	2.31E-5	1.999	7.10E-6	1.999	1.11E-5	1.999

Table 4: 3rd order quadtratic SVwith CFL=0.4 (periodic boundary condition)

	Local SV Formulation				SPSV Formulation			
H	L1 Eerror	L1 Order	L_∞error	L_∞order	L1 Eerror	L1 Order	L_∞error	L_∞order
10	4.93E-4		1.15E-3		3.90E-4		8.60E-4	
20	6.09E-5	3.0163	1.42E-4	3.011	4.26E-5	3.195	9.78E-5	3.136
40	7.57E-6	3.008	1.78E-5	3.000	5.14E-6	3.049	1.19E-5	3.031
80	9.45E-7	3.002	2.22E-6	3.000	6.35E-7	3.017	1.48E-6	3.007
160	1.18E-7	3.000	2.78E-7	3.000	7.90E-8	3.006	1.85E-7	3.001
320	1.47E-8	3.000	3.48E-8	2.999	9.86E-9	3.002	2.32E-8	3.000

Table 5: 2nd order linear SVwith CFL=0.6(Dirichlet boundary condition)

	Local SV Formulation				SPSV Formulation			
H	L1 Eerror	L1 Order	L_∞error	L_∞order	L1 Eerror	L1 Order	L_∞error	L_∞order
10	1.52E-2		7.58E-2		6.22E-3		9.82E-3	
20	3.91E-3	1.967	3.92E-2	0.949	1.73E-3	1.843	2.72E-3	1.852
40	9.81E-4	1.994	1.97E-2	0.991	4.49E-4	1.949	7.05E-4	1.947
80	2.45E-4	1.999	9.88E-3	0.998	1.13E-4	1.986	1.78E-4	1.986
160	6.13E-5	1.999	4.94E-3	0.999	2.84E-5	1.996	4.46E-5	1.996
320	1.53E-5	1.999	2.47E-3	0.999	7.10E-6	1.999	1.11E-5	1.999

Table 6: 3rd order quadratic SV with CFL=0.5 (Dirichlet boundary condition)

	Local SV Formulation				SPSV Formulation			
H	L1 Eerror	L1 Order	L_∞error	L_∞order	L1 Eerror	L1 Order	L_∞error	L_∞order
10	6.63E-4		3.49E-3		3.90E-4		8.60E-4	
20	7.04E-5	3.234	4.52E-4	2.952	4.26E-5	3.195	9.78E-5	3.136
40	8.16E-6	3.108	5.69E-5	2.989	5.14E-6	3.049	1.19E-5	3.031
80	9.82E-7	3.055	7.12E-6	2.997	6.35E-7	3.017	1.48E-6	3.007
160	1.20E-7	3.026	8.91E-7	2.999	7.90E-8	3.006	1.85E-7	3.001
320	1.49E-8	3.013	1.11E-7	2.999	9.86E-9	3.002	2.32E-8	3.000

From our numerical results, it is clear that the accuracy of the local SV formulation depends on the boundary conditions. For periodic boundary condition,$(k+1)$th order is achieved for k degree polynomial. Whereas, for Dirichlet boundary condition, $(k+1)$th order for L1 can be achieved. However, for L_∞ norm, it is not same for even and odd order. For even order, k^{th} order is achieved and for odd order, $(k+1)$th order is observed. Hence, we can conclude that with periodic boundary condition, only odd order SPSV formulation is more accurate than odd order local SV formulation. Whereas, for Dirichlet boundary condition, SPSV formulation is found to be more accurate than local SV formulation for all orders.

5.1.3. Stabilty Issue

Explicit scheme works well for quite small CFL number as compared to the implicit scheme. As we increse the the CFL number for an explicit scheme, its stability increases. We perform the numerical tests with varying CFL to check the respective stability of the formulations. The variation of maximum allowable CFL number versus the order of accuracy is shown in Figure 2.

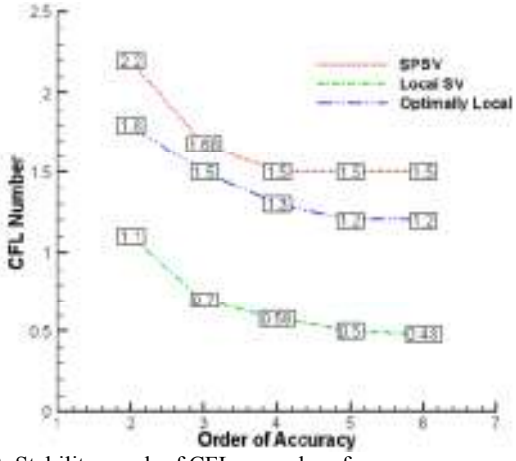

Fig. 2: Stability gragh of CFL vs. order of accuracy

Each formulation has almost the same behaviour for stabilty, but not the same value. From Figure 2, local SV formulation achieves stability at very small CFL number. Whereas, other formulations have relatively high threshold for the CFL number. Among them OASP formulation have high value of CFL number at the same respective order of accuracy and is the most stable among of the other formulations.

5.2. Burger Equation

Let us now consider one-dimansional Burger equation with implicit method, given by,

$$u_t + uu_x = \mu u_{xx}, \qquad x \in [0,1] \tag{22}$$

with initial condition $u(x,0) = 0$ and the boundary conditions $u(0,t) = 0$, $u(1,t) = -\tanh(\frac{1}{2\mu})$.

The problem has following exact steady solution,

$$u(x,t) = -\tanh(\frac{x}{2\mu}) \tag{23}$$

where $\mu = 0.1$. The simulation is run untill the convergence is achieved by using the implicit method. The convergence criterion is residuals $\leq 10^{-12}$ and the difference between two successive iterations is 10^{-15}. Deferred correction implicit method [23] is applied to achieve the numerical solution for SV method. Numerical results for the viscous Burger are given in Table 13-15.

The numerical results show that the local SV formulation is able to achieve $(k+1)$th order for k degree polynomial. Unlike diffusion equation, we observe even order for Burger equation. The reason for this behavior is the following. For diffusion equation, we observe the error on the left side and for Burger equation left hand side is almost the same. Therefore, no such behavior is observed. However, if we change the convection direction, the effects may be prominent.

Table 13: 2nd order linear SV

	Local SV Formulation				SPSV Formulation			
H	**L1 Error**	**L1 Order**	**L_∞Error**	**L_∞Order**	**L1 Error**	**L1 Order**	**L_∞Error**	**L_∞Order**
10	3.67E-4		3.08E-3		2.88E-4		3.79E-3	
20	4.00E-5	3.196	4.21E-4	2.874	2.47E-5	3.541	2.47E-4	3.941
40	4.93E-6	3.020	5.38E-5	2.968	2.99E-6	3.044	2.58E-5	3.258
80	6.19E-7	2.995	6.76E-6	2.991	3.77E-7	2.987	3.08E-6	3.067
160	7.76E-8	2.995	8.47E-7	2.997	4.75E-8	2.989	3.79E-7	3.020
320	9.72E-9	2.996	1.05E-7	2.999	5.98E-9	2.991	4.72E-8	3.6
640	1.29E-9	2.912	1.37E-8	2.949	7.74E-010	2.948	5.89E-9	3.002

Table 14: 3rd order quadratic SV

	Local SV Formulation				SPSV Formulation			
H	**L1 Error**	**L1 Order**	**L_∞Error**	**L_∞Order**	**L1 Error**	**L1 Order**	**L_∞Error**	**L_∞Order**
10	2.37E-3		2.00E-2		2.13E-3		6.65E-3	
20	7.03E-4	1.755	5.63E-3	1.829	8.08E-4	1.645	2.12E-3	1.397
40	1.94E-4	1.856	1.50E-3	1.909	2.43E-4	1.731	6.25E-4	1.765
80	5.11E-5	1.926	3.91E-4	1.938	6.60E-5	1.881	1.64E-4	1.927
160	1.31E-5	1.962	9.97E-5	1.971	1.71E-5	1.944	4.21E-5	1.964
320	3.32E-6	1.980	2.51E-5	1.986	4.36E-6	1.974	1.06E-5	1.979
640	8.36E-7	1.990	6.32E-6	1.993	1.10E-6	1.987	2.68E-6	1.990

6. CONCLUSION

In this paper, we have proposed a novel formulation for the spectral volume method referred to as Symmetry Preservation in Spectral Volume (SPSV) which is able to preserve the symmetry property of the elliptical operator. We implemented the newly proposed formulation on diffusion equation and Burger equation using two different boundary conditions, i.e., periodic and Dirichlet. We set three different criterion to evaluate this formulation: (i) symmetry preserving capability, (ii) accuracy, and (iii) stability. Furthermore, we also presented a detailed comparison between the local SV and the SPSV formulations. We showed that the optimal accuracy can be achieved in L_∞ norm with Dirichlet boundary conditions using SPSV formulation, which is not possible in the local SV formulation. Our comparison results show that the SPSV formulation is more accurate and stable.

REFERENCES

[1] Wang ZJ. Spectral (finite) volume method for conservation laws on unstructured grids: basic formulation. Journal of Computational Physics 2002; 178:210-251.

[2] Wang ZJ, Liu Y. Spectral (finite) volume method for conservation laws on unstructured grids III: one-dimensional systems and partition optimization. Journal of Scientific Computing 2004; 20:137-157.

[3] Wang ZJ, Zhang L, Liu Y. Spectral (finite) volume method for conservation laws on unstructured grids IV: extension to two-dimensional Euler equations. Journal of Computational Physics 2004; 194:716-741.

[4] Cockburn B, Lin S-Y, Shu C-W. TVD Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws III: one-dimensional systems. Journal of Computational Physics 1989; 84: 90-113.

[5] Zhang M, Shu C-W. An analysis of and a comparison between the discontinuous Galerkin and spectral finite volume methods. Computers and Fluids,2005;34: 581-592.

[6] Sun Y, Wang ZJ. Evaluation of discontinuous Galerkin and spectral volume methods for scalar and system conservation laws on unstructured grids. International Journal for Numerical Methods in Fluids 2004; 45: 819-838.

[7] Barth TJ, Frederickson PO. High-order solution of the Euler equations on unstructured grids using quadratic reconstruction. AIAA Paper No. 90-0013, 1990.

[8] Abgrall, R. (1994). On essentially non-oscillatory schemes on unstructured meshes: analysis and implementation. J. Comp. Phys. 114, 45-58.

[9] Friedrich, O. (1998). Weighted essentially non-oscillatory schemes for the interpolation of mean values on unstructured grids. J. Comp. Phys. 144, 194-212.

[10] Harten, A., Engquist, B., Osher, S., and Chakravarthy, S. (1987). Uniformly high order essentially non-oscillatory schemes III. J. Comp. Phys. 71, 231.

[11] Hu, C., and Shu, C. W. (1999). Weighted essentially non-oscillatory schemes on triangular meshes. J. Comp. Phys. 150, 97-127.

[12] Shu C-W. Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws. In Advanced Numerical Approximation of Nonlinear Hyperbolic Equations, Cockburn B, Johnson C, Shu C-W, Tadmor E, Quarteroni A (eds). Lecture Notes in Mathematics, vol. 1697. Springer:Berlin,1998; 325-432.

[13] Godunov SK. A finite-difference method for the numerical computation of discontinuous solutions of the equations of uid dynamics. Mathematics of the USSR-Sbornik 1959; 47:271.

[14] Shu C-W, Osher S. Efficient implementation of essentially non-oscillatory shock capturing schemes. Journal of Computational Physics 1988; 77:439-471.

[15] Shu C-W. TVB uniformly high-order schemes for conservation laws. Mathematics of Computation 1987; 49:105-121.

[16] Yuzhi Sun, Z.J. Wang.Formulations and analysis of the spectral volume method for the diffusion equation. Communication in Numerical Methods in Engineering 2004;20:927-937

[17] Cockburn B, Shu C-W. The local discontinuous Galerkin method for time-dependent convection di usion system.SIAM Journal on Numerical Analysis 1998; 35:2440-2463.

[18] Baumann CE, Oden JT. A discontinuous hp finite element method for convection-diffusion problems. Computer Methods in Applied Mechanics and Engineering 1999; 175:311-341.

[19] Oden JT, Babuska I, Baumann CE. A discontinuous hp finite element method for diffusion problems. Journal of Computational Physics 1998; 146:491-519.

[20] Riviere B, Wheeler M, Girault V. A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. SIAM Journal on Numerical Analysis 2001; 39:902-931.

[21] D.A. Kopriva. A conservation staggered-grid multidomain spectral method for the compressible Navier-Stokes equations. Journal of Computational Physics 1998;143:125-158. 39:902-931.

[22] Y. Sun, Z.J. Wang and Y. Liu. High-order multidomain spectral difference method for the Navier-Stokes equations. AIAA-2006-0301.

[23] Minion, Michael L. Semi-implicit spectral deferred correction methods for ordinary differential equations. Communications in Mathematical Sciences 1.3 (2003): 471-500.